FAMILIES OF CYCLES

DAVID RYDH

Abstract. Let X be an algebraic space, locally of finite type over an *arbitrary* scheme S. We give a definition of *relative cycles* on X/S. When S is reduced and of characteristic zero, this definition agrees with the definitions of Barlet, Kollár and Suslin-Voevodsky when these are defined. Relative multiplicity-free cycles and relative Weil-divisors over arbitrary parameter schemes are then studied more closely. We show that relative *normal* cycles are given by flat subschemes, at least in characteristic zero. In particular, the morphism $\mathrm{Hilb}^{\mathrm{equi}}_r(X/S) \to \mathrm{Chow}_r(X/S)$ taking a subscheme, equidimensional of dimension r, to its relative fundamental cycle of dimension r, is an isomorphism over normal subschemes.

When S is of characteristic zero, any relative cycle induces a unique relative fundamental class. The set of Chow classes in the sense of Angéniol constitute a subset of the classes corresponding to relative cycles. When α is a relative cycle such that either S is reduced, α is multiplicity-free, or α is a relative Weil-divisor, then its relative fundamental class is a Chow class. In particular, the corresponding Chow functors agree in these cases.

Introduction

The Chow variety $\mathrm{ChowVar}_{r,d}(X \hookrightarrow \mathbb{P}^n)$, parameterizing families of cycles of dimension r and degree d on a projective variety X, was constructed in the first half of the twentieth century [CW37, Sam55]. The main goal of this paper is to define a natural contravariant functor $\mathrm{Chow}_{r,d}(X)$ from schemes to sets, such that its restriction to reduced schemes is represented by $\mathrm{ChowVar}_{r,d}(X)$. Here $\mathrm{ChowVar}_{r,d}(X)$ is a reduced variety coinciding with $\mathrm{ChowVar}_{r,d}(X \hookrightarrow \mathbb{P}^n)$ for a sufficiently ample projective embedding $X \hookrightarrow \mathbb{P}^n$ [Hoy66]. In characteristic zero, the Chow variety $\mathrm{ChowVar}_{r,d}(X \hookrightarrow \mathbb{P}^n)$ is independent on the embedding [Bar75] but this is not the case in positive characteristic [Nag55].

We will first define a notion of *relative cycles* on X/S. This definition is given in great generality without any assumptions on S and only assuming that X/S is locally of finite type. This definition includes non-equidimensional and even non-separated relative cycles. We then let

$$Cycl(X/S) = \{\text{relative cycles on } X/S\}$$
$$\mathrm{Chow}_r(X/S)(T) = \left\{\begin{matrix}\text{proper relative cycles which are equi-}\\ \text{dimensional of dimension } r \text{ on } X \times_S T/T\end{matrix}\right\}.$$

Date: 2008-05-18.

2000 *Mathematics Subject Classification.* Primary 14C05; Secondary 14C25.

Key words and phrases. Chow scheme, Hilbert scheme, divided powers, cycles, Weil divisors.

If X is projective, the functor $\mathrm{Chow}_r(X)$ is a disjoint union of the subfunctors $\mathrm{Chow}_{r,d}(X)$ parameterizing cycles of a fixed degree. We also let

$$\mathrm{Chow}(X)(T) = \{\text{proper equidimensional relative cycles on } X \times_S T/T\}.$$

A similar Chow-functor, which we will denote by $\mathrm{Ang}_r(X)$, has been constructed by Angéniol [Ang80] in characteristic zero and we will present some evidence indicating that Chow = Ang in characteristic zero. In fact, there is a natural monomorphism $\mathrm{Ang}_r(X) \to \mathrm{Chow}_r(X)$ which is a bijection when restricted to reduced T and between the open subfunctors parameterizing multiplicity-free cycles.

It is known [Bar75, Gue96, Kol96, SV00] that if T is a normal scheme of *characteristic zero*, then there is a one-to-one correspondence between T-points of the Chow variety and cycles $\mathcal{Z}$ on $X \times T$ which are equidimensional of relative dimension r and whose generic fiber has degree d. Thus when S is normal and of characteristic zero we define a relative r-cycle on X/S to be a cycle on X which is equidimensional of relative dimension r and the definition of $\mathrm{Chow}_r(X)(T)$ for T normal follows. There is a subtle point here concerning the pull-back $\mathrm{Chow}_r(X)(T) \to \mathrm{Chow}_r(X)(T')$ for a morphism $T' \to T$ between normal schemes. If $t \in T$ is a point, then the naïve fiber $\mathcal{Z}_t$ does not necessarily coincide with the cycle corresponding to the morphism $\mathrm{Spec}(k(t)) \to T \to \mathrm{ChowVar}_{r,d}(X)$.

This problem is due to the fact that $\mathcal{Z}$ is not "flat" over T. As an illustration, let $T = \mathrm{Spec}(k[s,t])$ be the affine plane and consider the family of zero-cycles on $X = \mathrm{Spec}(k[x,y])$ of degree two given by the primitive cycle $\mathcal{Z} = [Z]$ where $Z \hookrightarrow X \times T$ is the subvariety given by the ideal $(x^2 - s^3t, y^2 - st^3, xy - s^2t^2, tx - sy)$. On the open subset $T \setminus (0,0)$, we have that Z is flat of rank two, but the fiber over the origin is the subscheme defined by (x^2, y^2, xy) which has rank three. The naïve fiber in this case would be three times the origin of X while the correct fiber is two times the origin.

If T is a smooth curve, the above "pathology" does not occur as then every cycle is flat. If T is a smooth variety, the correct fiber $\mathcal{Z}_t$ can be defined through intersection theory [Ful98, Ch. 10]. If T is a normal variety, then the correct fiber $\mathcal{Z}_t$ can be defined through Samuel multiplicities [SV00, Thm. 3.5.8]. For an arbitrary reduced scheme T, the fiber of a cycle $\mathcal{Z}$ on $X \times T$ at t can be defined by taking the "limit cycle" along a curve passing through t as defined by Kollár [Kol96] and Suslin-Voevodsky [SV00]. This construction may depend on the choice of the curve, but if T is normal and of characteristic zero the limit cycle is well-defined. If T is weakly normal, then $\mathcal{Z}$ will be a relative cycle if and only if the limit cycle is well-defined for every point $t \in T$. In positive characteristic, even if T is normal, the limit cycle may have rational coefficients [SV00, Ex. 3.5.10].

It is natural to include cycles with certain rational coefficients in positive characteristic [Ryd08b] and we will call these cycles *quasi-integral*. A relative cycle over a *perfect* field will always have integral coefficients. The denominator of the multiplicity of a subvariety is bounded by its *inseparable discrepancy*.

The limit-cycle condition only gives the correct functor for semi-normal schemes. The problem is easily illustrated by taking a geometrically unibranch but non-normal parameter scheme S, such as a cuspidal curve. The normalization $X \to S$ then satisfies the limit-cycle condition — the limit-cycle of the singular point of S is the corresponding point of X with multiplicity one. We thus obtain a "relative" zero-cycle of degree one $X \to S$ but this does not correspond to a morphism $S \to \mathrm{ChowVar}_{0,1}(X) \cong X$.

Definition of the Chow functor. The definition of the Chow functor is based upon the assumption that $\mathrm{Chow}_{0,d}(X)$ should be represented by the scheme of divided powers $\Gamma^d(X/S)$. This is in agreement with the conditions on $\mathrm{Chow}_{0,d}(X)$ imposed at the beginning as $\Gamma^d(X/S)_{\mathrm{red}} = \mathrm{ChowVar}_{0,d}(X/S)$, cf. [Ryd08c]. If X/S is flat or the characteristic of S is zero, then $\Gamma^d(X/S) = \mathrm{Sym}^d(X/S)$ [Ryd08a].

We let $\Gamma^\star(X/S) = \coprod_{d\geq 0} \Gamma^d(X/S)$ which thus represents $\mathrm{Chow}_0(X)$. A relative proper zero-cycle on X/S is then a morphism $S \to \Gamma^\star(X/S)$. If S is reduced, a relative proper zero-cycle is represented by a quasi-integral cycle on X, such that its support is proper and equidimensional of dimension zero over S. For a reduced scheme S, we then make the following definition of a higher-dimensional relative cycle:

If S is reduced, then a relative cycle on X/S of dimension r is a cycle $\mathcal{Z}$ on X which is equidimensional of dimension r over S and such that for any *smooth projection* (U, B, T, p, g, φ) consisting of a diagram

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d, "\varphi"] & X \arrow[dd] \\ B \arrow[d] \arrow[dr, phantom, "\circ"] & \\ T \arrow[r, "g"] & S \end{tikzcd}$$

where $U \to X$, $B \to T$ and $T \to S$ are smooth, $U \to X \times_S T$ is étale, and $\varphi : U \to B$ is finite over the support of $p^*\mathcal{Z}$, we have that $p^*\mathcal{Z}$ is a relative (proper) zero-cycle over B, i.e., given by a morphism $B \to \Gamma^\star(U/B)$.

We will show that when S is of characteristic zero, the above definition agrees with the definition given by Barlet [Bar75] in the complex-analytic case and the definition given by Angéniol [Ang80] in the algebraic setting. The functor $\mathrm{Chow}_r(X)_{\mathrm{red}}$ is then an algebraic space which coincides with $\mathrm{ChowVar}_r(X)$ when X is projective. Over semi-normal schemes, we recover the definitions with limit-cycles due to Kollár and Suslin-Voevodsky.

Definition over arbitrary parameter schemes. Over a general scheme S it is more complicated to define what a relative cycle on X/S is. A main obstacle is the fact that relative cycles on X/S are not usually represented by cycles on X. The course taken by El Zein and Angéniol [AEZ78, Ang80] is to represent a relative cycle by its *relative fundamental class*. This is a cohomology class in $c \in \mathrm{Ext}^{-r}_{Z_m}(\Omega^r_{Z_m/S}, \mathcal{D}^\bullet_{Z_m/S})$, where $j_m : Z_m \hookrightarrow X$ is an infinitesimal neighborhood of the support Z of the relative cycle. Such a cohomology class induces, by duality, a class in $\mathrm{Ext}_X^{-r}((j_m)_*\mathcal{O}_{Z_m}\otimes\Omega^r_{X/S},\mathcal{D}^\bullet_{X/S})$ and, if X/S is smooth, a class in $\mathrm{H}_Z^{-r}(X,(\Omega^r_{X/S})^\vee\otimes\mathcal{D}^\bullet_{X/S})=\mathrm{H}_Z^{n-r}(X,\Omega^{n-r}_{X/S})$.

The connection with cycles is as follows. A class c, supported at $Z\subseteq X$, in one of the above cohomology groups, induces for any projection (U,B,T,p,g,φ) with $U\to X$, $B\to T$ and $T\to S$ smooth and $\varphi: U\to B$ finite over $p^{-1}(Z)$, a trace homomorphism $\mathrm{tr}(c):\varphi_*\Omega^r_{p^{-1}(Z_m)/T}\to\Omega^r_{B/T}$. This homomorphism extends uniquely to a homomorphism $\mathrm{tr}(c):\varphi_*\Omega^\bullet_{p^{-1}(Z_m)/T}\to\Omega^\bullet_{B/T}$, commuting with the differentials, and in particular we obtain a trace map $\mathrm{tr}(c):\varphi_*\mathcal{O}_{p^{-1}(Z_m)}\to\mathcal{O}_B$. In characteristic zero, a family of zero-cycles $B\to\Gamma^d(p^{-1}(Z_m)/B)$ is determined by its trace $\varphi_*\mathcal{O}_{p^{-1}(Z_m)}\to\mathcal{O}_B$.

In characteristic zero, Angéniol [Ang80, Thm. 1.5.3] gives a condition characterizing the homomorphisms $\varphi_*\mathcal{O}_{p^{-1}(Z_m)}\to\mathcal{O}_B$ which are the traces of families $B\to\Gamma^d(p^{-1}(Z_m)/B)$. He then generalizes this condition to a condition on $\mathrm{tr}(c):\varphi_*\Omega^\bullet_{p^{-1}(Z_m)/T}\to\Omega^\bullet_{B/T}$ which is stable under the choice of projection [Ang80, Prop. 2.3.5]. Thus, if $\mathrm{tr}(c)$ satisfies this condition, then the induced trace for any projection comes from a family of zero-cycles. It is not clear whether the converse is true, i.e., if a class such that the induced trace on any projection comes from a family of zero-cycles satisfies Angéniol's condition.

In positive characteristic, some kind of "crystalline duality" would be required to accomplish a similar description and we do not follow this line. Our definition is more straight-forward. We define a relative cycle, supported on a subset $Z\subseteq X$, to be a collection of relative zero-cycles $B\to\Gamma^\star(U/B)$ for every projection (U,B,Z) of X/S such that $p^{-1}(Z)\to B$ is finite. We further impose natural compatibility conditions on the zero-cycles of different projections. At first glance, this looks impractical to work with but we describe situations in which the relative cycles are easier to describe.

- (A1) If S is reduced, then every relative cycle is induced by an ordinary cycle on X as described above, cf. Corollary (8.7).
- (A2) If α is a multiplicity-free relative cycle on X/S, i.e., if the pull-back cycles α_s are without multiplicities for every geometric point $s\to S$, then α is induced by a subscheme of X which is flat over a fiberwise dense subset, cf. Corollary (9.9).
- (A3) If α is a relative Weil-divisor on X/S, i.e., if X/S is equidimensional of dimension $r+1$, and if X/S is flat with (R_1)-fibers, e.g., if X/S has normal fibers, then α is induced by a subscheme of X which is a relative Cartier divisor over a fiberwise dense subset, cf. Corollary (9.16).

In positive characteristic, the descriptions (A2) and (A3) are unfortunately conjectural except when S is reduced. Note that in these three cases we obtain an object which represents the cycle α but not all such objects induce a relative cycle. There are however correspondences as follows:

- (B1) S *normal.* Relative cycles over S correspond to effective quasi-integral cycles on X with universally open support, cf. Theorem (10.1).

(B2) S *semi-normal*. Relative cycles over S correspond to ordinary cycles on X with universally open support such that the limit-cycle for every point $s \in S$ is well-defined and quasi-integral, cf. Theorem (10.17).
(B3) S arbitrary. *Smooth* relative cycles correspond to subschemes which are smooth, cf. Theorem (9.8).
(B4) S arbitrary. *Normal* relative cycles correspond to subschemes which are flat with normal fibers, cf. Theorem (12.8).
(B5) S arbitrary, X/S smooth. *Relative Weil-divisors* on X/S correspond to relative Cartier-divisors on X/S, cf. Theorem (9.15).
(B6) S arbitrary, X/S flat with geometrically (R_2)-fibers. *Relative Weil-divisors* on X/S correspond to Weil-divisors Z on X/S such that Z is a relative Cartier-divisor over an open subset of Z containing all points of relative codimension at most one, cf. Theorem (11.7).
(B7) S arbitrary. *Multiplicity-free* relative cycles which are geometrically (R_1) correspond to subschemes which are flat with geometrically (R_1)-fibers over an open subset containing all points of relative codimension at most one, cf. Theorem (11.5).
(B8) S reduced, X/S flat with geometrically (R_1)-fibers. *Relative Weil-divisors* on X/S correspond to Weil-divisors Z on X/S which are relative Cartier-divisors over an open fiberwise dense subset of Z, cf. Theorem (11.7).
(B9) S reduced. *Multiplicity-free* relative cycles correspond to subschemes which are flat with reduced fibers over an open fiberwise dense subset, cf. Theorem (11.5).

Here (B3)–(B7) are conjectural in positive characteristic.

In characteristic zero, it follows from Bott's theorem on grassmannians, similarly as in [AEZ78], that a relative cycle induces a fundamental class c in one of the cohomology groups discussed above. It is however not clear that c satisfies the conditions imposed by Angéniol except for relative cycles as in (A1)–(A3).

Push-forward and pull-back. If $f : X \to Y$ is a *finite* morphism of S-schemes, then there is a natural functor — the push-forward — from relative cycles on X/S to relative cycles on Y/S. When S is reduced, the push-forward of relative cycles coincides with the ordinary push-forward of cycles.

It is reasonable to believe that for any proper morphism $f : X \to Y$ there should be a push-forward functor $f_* : \mathit{Cycl}(X/S) \to \mathit{Cycl}(Y/S)$ coinciding with the ordinary push-forward of cycles when S is reduced. Recall that if V is a subvariety of X, then the push-forward $f_*([V])$ of the cycle $[V]$ is $\deg\big(k(V)/k(f(V))\big)[f(V)]$ if $f|_V$ is generically finite and zero otherwise. The push-forward for arbitrary cycles is then defined by linearity. If α is a relative cycle on X/S then it is straight-forward to define $f_*\alpha$ on a dense subset of its support, but the verification that this cycle extends to a cycle, necessarily unique, on the whole support is only accomplished in the cases (A1) and (B1)–(B9) above using flatness.

If $f : U \to X$ is a flat morphism which is equidimensional of dimension r, then it is again reasonable that there should be a pull-back functor $f^* :$

$\mathit{Cycl}(X/S) \to \mathit{Cycl}(Y/S)$ coinciding with the flat pull-back of cycles when S is reduced. Contrary to the case with the push-forward it is not even clear how f^* should be defined in general. It is possible to partially define the pull-back by giving families of zero-cycles on certain projections. In the cases (A1)–(A3), it is clear how the pull-back should be defined generically on any projection but it is only in the cases (B1)–(B9) that it is shown that the generic pull-back extends to a relative cycle.

If $f : U \to X$ is *smooth* of relative dimension r, then it is possible to construct a pull-back $f^*(c)$ for the cohomological description of relative cycles in characteristic zero. We will show that smooth pull-back exists when S is reduced but in general this is as problematic as the flat pull-back. This motivates the following alternative definition of relative cycles. A relative cycle α on X/S with support Z consists of relative zero-cycles $\alpha_{U/B}$ on U/B for any commutative square

$$\begin{array}{ccc} U & \xrightarrow{p} & X \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & X \end{array}$$

such that p and g are smooth and $p^{-1}(Z) \to B$ is finite. These zero-cycles are required to satisfy natural compatibility conditions. Every relative cycle of the new definition determines a unique relative cycle of the first definition. With the new definition, it is at least clear that smooth pull-backs exist.

Products and intersections. If α and β are relative cycles on X/S and Y/S respectively, then it is reasonable to demand that there should be a natural relative cycle $\alpha \times \beta$ on $X \times_S Y/S$. This relative cycle is only defined under the same conditions as the flat pull-back.

If α is a relative cycle on X/S and D is a relative Cartier divisor on X/S meeting α properly in every fiber, then there is a relative cycle $D \cap \alpha$ on X/S. If two relative cycles α and β on a smooth scheme X/S meet properly in every fiber, then under the assumption that $\alpha \times \beta$ is defined, we then define $\alpha \cap \beta$ as the intersection of $\alpha \times \beta$ with the diagonal $\Delta_{X/S}$.

Overview of contents. The paper is naturally divided into three parts. In the first part, Sections 1–6, we give the foundations on relative cycles. In the second part, Sections 7–12, we treat relative cycles which are flat, multiplicity-free, normal or smooth, relative Weil divisor and relative cycles over reduced parameter schemes. In the third part, Sections 13–17, we discuss proper push-forward, flat pull-back and intersections of cycles, compare our definition of relative cycles with Angéniol's definition and mention the classical construction of the Chow variety via Grassmannians. The third part is very brief and many results are only sketched.

In Section 1, we briefly recall the results on proper relative zero-cycles from [Ryd08a, Ryd08b]. We also show that the definition of a proper relative zero-cycle on X/S is local on X with respect to finite étale coverings.

In Section 2, we define *non-proper* relative zero-cycles. This is done by working étale-locally on the carrier scheme X. A non-proper relative zero-cycle is a proper relative zero-cycle if and only if its support is proper. This gives a new étale-local definition of proper relative zero-cycles.

In Section 3, a topological condition (T) on morphisms is introduced. This condition is closely related to open morphisms. In fact, if S is locally noetherian and $f : X \to S$ is locally of finite type, then f is universally open if and only if it is universally (T). Universally open morphisms and equidimensional morphisms satisfy (T).

In Section 4, we define higher-dimensional relative cycles. A priori, the support of a higher-dimensional relative cycle only satisfies (T), but we show that the support is universally open.

In Section 5, we show that in the definition of a relative cycle, it is enough to consider *smooth* projections. We then briefly discuss how the definition of a relative cycle can be modified so that pull-back by smooth morphisms exist.

In Section 6, conditions for when a relative cycle on an open subset $U \subseteq X$ extends to a relative cycle on X are given. We also give a slightly generalized version of Chevalley's theorem on universally open morphisms.

In Section 7, we show that any flat and finitely presented sheaf $\mathcal{F}$ induces a relative cycle, the *norm family*. We thus obtain morphisms from the Hilbert and the Quot functors to the Chow functor.

In Section 8, we associate an ordinary cycle $\mathrm{cycl}(\alpha)$ on X to any relative cycle α on X/S. If S is reduced, this cycle uniquely determines α. This is (A1).

In Section 9, we show that smooth relative cycles correspond to smooth subschemes and that relative Weil divisors on smooth carrier schemes correspond to relative Cartier divisors. This is (B3) and (B5). Assuming only generic smoothness, we obtain the descriptions (A2) and (A3). These results are only shown in characteristic zero but are presumably valid in arbitrary characteristic.

In Section 10, we study relative cycles over reduced parameter schemes and obtain the characterizations (B1) and (B2). We also describe the pull-back of cycles via Samuel multiplicities.

In Section 11, we introduce n-flat and n-smooth morphisms and give the characterizations (B6)–(B9). In Section 12, we prove a generalized Hironaka lemma. Together with (B7), this result yields (B4). In particular, the Hilb-Chow morphism is an isomorphism over the locus parameterizing normal subschemes. All these results depend upon (B3) but are otherwise characteristic-free.

In Sections 13–15, proper push-forward, flat pull-back and intersections of relative cycles are discussed. In Section 16, we indicate the existence of a relative fundamental class to any relative cycle and compare our functor with Angéniol's and Barlet's functors. Finally, in Section 17, we discuss the incidence correspondence and the classical Chow-construction.

In the Appendix, an overview of duality and (relative) fundamental classes is given.

Terminology and assumptions. As families of cycles are defined étale-locally, the natural choice is to use algebraic spaces instead of schemes. In fact, all results are true for algebraic spaces. For convenience, we only treat relative cycles on X/S where S is a scheme, but this is no restriction as the definition is étale-local on both S and X.

We allow relative cycles to have *non-closed* support. The reason for this is that if α is a relative cycle on X/S then it decomposes as a sum $\alpha_0 + \alpha_1 + \dots + \alpha_r$ where α_i is supported on the locally closed subset consisting of points of relative dimension i. It is likely that the assumption that a relative cycle has closed support is missing in some statements and the reader may choose to assume that all relative cycles have closed support (except in the example above).

Usually, a cycle is a finite formal sum of equidimensional closed subvarieties. As we treat relative cycles which are not equidimensional and also not necessarily closed, we make the following definition. A *cycle* α on X is a locally closed subset $Z \subseteq X$ together with rational numbers (m_{Z_i}) indexed by the irreducible components $\{Z_i\}$ of Z. The irreducible sets $\{Z_i\}$ are the *components* of α and the numbers (m_{Z_i}) are the *multiplicities* of α. When the m_{Z_i}'s are integers, we say that α is *integral*. When the m_{Z_i}'s are non-negative, we say that α is *effective*. Every cycle is uniquely represented as a formal sum $\sum_i m_{Z_i}[Z_i]$. This sum is locally finite if X is locally noetherian. Note that this definition excludes cycles with embedded components. Through-out this paper we will only consider effective cycles.

Let X/S be an algebraic space locally of finite type. We say that $x \in X$ has *relative codimension* n if the codimension of $\overline{\{x\}}$ in its fiber X_s is n. A useful fact is that if $X \to B$ is a quasi-finite morphism, $B \to S$ is smooth and $x \in X$ has relative codimension n over a point of depth m in S, then its image $b \in B$ has depth $n + m$. This is why the characterizations (A2)–(A3) and (B6)–(B9) only involves the codimension.

Noetherian assumptions are eliminated in many instances but often only with a brief sketch in the proof. Sometimes we use the notion of associated points on a non-noetherian schemes. In the terminology of Bourbaki these are the points corresponding to *weakly* associated prime ideals. These satisfy the usual properties of associated points, e.g., an open (retro-compact) subset $U \subseteq X$ is schematically dense if and only if U contains all associated points. Recall that on a locally noetherian scheme X, a point $x \in X$ is associated if and only if X has depth zero at x. In the non-noetherian case the condition that S has (locally) a finite number of irreducible components appears frequently.

Contents

1. Proper relative zero-cycles

We recall the main results of [Ryd08a, Ryd08b], here stated only for schemes locally of finite type. We then show that the definition of proper relative zero-cycles on X/S is étale-local on X. At the end, the underlying cycle of a relative cycle is briefly discussed.

Definition (1.1). Let $f : Z \to S$ be affine. Then we let $\Gamma^d(Z/S)$ be the spectrum of the algebra of divided powers $\Gamma^d_{\mathcal{O}_S}(f_*\mathcal{O}_Z)$.

Definition (1.2). Let X/S be a separated algebraic space locally of finite type. A *relative zero-cycle of degree d* on X consists of a closed subscheme $Z \hookrightarrow X$ such that $Z \hookrightarrow X \to S$ is finite, together with a morphism $\alpha : S \to \Gamma^d(Z/S)$. Two relative zero-cycles (Z_1, α_1) and (Z_2, α_2) are *equivalent* if there is a closed subscheme Z of both Z_1 and Z_2 and a morphism $\alpha : S \to \Gamma^d(Z/S)$ such that α_i is the composition of α and the morphism $\Gamma^d(Z/S) \hookrightarrow \Gamma^d(Z_i/S)$ for $i = 1, 2$.

If $g : S' \to S$ is a morphism of spaces and (Z, α) is a relative cycle on X/S, we let $g^*(Z, \alpha) = \bigl(g^{-1}(Z), g^*\alpha\bigr)$ be the pull-back along g.

If (Z, α) is a relative zero-cycle, then there is a unique minimal closed subscheme $\mathrm{Image}(\alpha) \hookrightarrow Z$ such that (Z, α) is equivalent to a relative zero-cycle $(\mathrm{Image}(\alpha), \alpha')$. The subscheme $\mathrm{Image}(\alpha)$ is called the *image* of α and its reduction $\mathrm{Supp}(\alpha) := \mathrm{Image}(\alpha)_{\mathrm{red}}$ is the *support* of α. The image commutes with smooth base change but not with arbitrary base change. The support commutes with arbitrary base change in the sense that $\mathrm{Supp}(\alpha \times_S S') = (\mathrm{Supp}(\alpha) \times_S S')_{\mathrm{red}}$.

Notation (1.3). If $\alpha : S \to \Gamma^d(X/S)$ is a morphism then we will by abuse of notation often write α for the first map in the canonical factorization $S \to \Gamma^d(\mathrm{Image}(\alpha)/S) \hookrightarrow \Gamma^d(X/S)$ of α.

Definition (1.4). Let X/S be a separated algebraic space locally of finite type. We let $\underline{\Gamma}^d_{X/S}$ be the contravariant functor from S-schemes to sets defined as follows. For any S-scheme T we let $\underline{\Gamma}^d_{X/S}(T)$ be the set of equivalence classes of relative zero cycles (Z,α) of degree d on $X \times_S T/T$. For any morphism $g : T' \to T$ of S-schemes, the map $\underline{\Gamma}^d_{X/S}(g)$ is the pull-back of relative cycles as defined above.

An element of $\underline{\Gamma}^d_{X/S}(T)$ will be called a *family of zero cycles of degree d* on X/S parameterized by T. By abuse of notation, henceforth a relative cycle will always denote an equivalence class of relative cycles. The main result of [Ryd08a] is that $\underline{\Gamma}^d_{X/S}$ is representable by a separated algebraic space $\Gamma^d(X/S)$ — the scheme of divided powers — which coincides with the scheme in Definition (1.1) when X/S is affine. If X is locally of finite type (resp. locally of finite presentation) over S then so is $\Gamma^d(X/S)$.

Definition (1.5) ([Ryd08b, Def. 2.1]). Let X/S be a separated scheme (or algebraic space), locally of finite type over S. We let $\Gamma^\star(X/S) = \coprod_{d\geq 0}\Gamma^d(X/S)$. A *proper family of zero-cycles* on X/S parameterized by T is a morphism $\alpha : T \to \Gamma^\star(X/S)$. A *proper relative zero-cycle* on X/S is a morphism $\alpha : S \to \Gamma^\star(X/S)$. If the image of a point $s \in S$ by α lies in $\Gamma^d(X/S)$ then we say that α has degree d at s.

If $X = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(A)$ are affine, then $\Gamma^\star(X/S)$ represents multiplicative laws which are not necessarily homogeneous [Ryd08b, Thm. 2.3]. To be precise, if $T = \operatorname{Spec}(A')$ then $\operatorname{Hom}_S(T, \Gamma^\star(X/S))$ is the set of multiplicative laws $B \to A'$.

We will use the following results and constructions from [Ryd08a, Ryd08b]:

(i) If $f : X \to Y$ is a morphism, then the *push-forward* along f is the morphism $f_* : \Gamma^d(X/S) \to \Gamma^d(Y/S)$ taking a family (Z,α) onto the family $(f_T(Z), f_*\alpha)$. Here $f_*\alpha$ is the composition of $\alpha : T \to \Gamma^d(Z/T)$ and $\Gamma^d(Z/T) \to \Gamma^d(f_T(Z)/T)$. The image does not commute with the push-forward in general, but the support does, i.e., $\operatorname{Supp}(f_*(\alpha)) = f_T(\operatorname{Supp}(\alpha))$ [Ryd08a, 3.3.7].

(ii) The image $\operatorname{Image}(\alpha) \hookrightarrow X \times_S T$ of a proper family of cycles $\alpha : T \to \Gamma^\star(X/S)$ is finite and universally open over T [Ryd08a, 2.4.6, 2.5.7].

(iii) If T is a reduced scheme, then the image $\operatorname{Image}(\alpha) \hookrightarrow X \times_S T$ of a family $\alpha : T \to \Gamma^\star(X/S)$ is reduced [Ryd08a, 2.4.6].

(iv) If k is an algebraically closed field, then there is a one-to-one correspondence between k-points of $\Gamma^d(X/S)$ and effective zero cycles of degree d on $X \times_S \operatorname{Spec}(k)$ [Ryd08a, 3.1.9].

(v) If $f : Z \to S$ is finite and flat of finite presentation, i.e., such that $f_*\mathcal{O}_Z$ is a locally free $\mathcal{O}_S$-module, then there is a canonical family $\mathcal{N}_{Z/S} : S \to \Gamma^\star(Z/S)$, the *norm* of f. The support of $\mathcal{N}_{Z/S}$ is Z_{red} but in general the image of $\mathcal{N}_{Z/S}$ can be smaller than Z. If f is in addition *étale* then $\operatorname{Image}(\mathcal{N}_{Z/S}) = Z$ and the image commutes with arbitrary base change [Ryd08b, Prop. 3.2]. More generally, if X/S is affine, then a norm family $\mathcal{N}_{\mathcal{F}/S} : S \to \Gamma^\star(X/S)$ is defined for any quasi-coherent sheaf $\mathcal{F}$ on X such that $f_*\mathcal{F}$ is locally free.

(vi) If α is a family of degree d parameterized by T such that for any algebraically closed field k and point $t : \operatorname{Spec}(k) \to T$ the support of

α_t has (at least) d points then we say that α is *non-degenerate.* Then $Z = \operatorname{Image}(\alpha)$ is étale of constant rank d and $\alpha = \mathcal{N}_{Z/T}$ [Ryd08b, Cor. 5.7].

(vii) If X/S is a smooth curve, then $\Gamma^d(X/S) \cong \operatorname{Hilb}^d(X/S)$, i.e., for any relative cycle α on X/S there is a unique subscheme $Z \hookrightarrow X$, flat and finite over S, such that $\mathcal{N}_Z = \alpha$ [Ryd08b, Prop. 5.8]. Note however that $\operatorname{Image}(\alpha)$ does not always equal Z.

(viii) Let U/X be a separated algebraic space. If α is a proper relative zero-cycle on X/S and β is a proper relative zero-cycle on U/X, then there is a proper relative zero-cycle $\alpha * \beta$ on U/S. If α and β have degrees d and e respectively, then $\alpha * \beta$ has degree de. If T is the spectrum of an algebraically closed field and α corresponds to the cycle $[x_1] + [x_2] + \cdots + [x_d]$, then $\alpha * \beta$ corresponds to the cycle $\beta_{x_1} + \beta_{x_2} + \cdots + \beta_{x_d}$ [Ryd08b, §7].

We will now show that proper relative zero-cycles of X/S can be defined étale-locally on X.

Definition (1.6). Let X/S be a separated algebraic space and let $f : U \to X$ be a separated and étale morphism. Let $\alpha : S \to \Gamma^\star(X/S)$ be a proper family of zero-cycles on X and assume that f is proper over the support of α. Then f is étale and finite over $Z = \operatorname{Image}(\alpha)$ and we let $f^*(\alpha) = \alpha * \mathcal{N}_{f|_Z}$ which we by push-forward consider as a family on U/S. When f is an open immersion, then we let $\alpha|_U = f^*\alpha$.

The notation $f^*(\alpha)$ is reasonable in view of the following results:

Proposition (1.7) ([Ryd08b, Prop. 7.5])**.** *Let X/S be a separated algebraic space and let α be a proper relative cycle on X/S. Let $f : S' \to S$ be a finite étale morphism and denote by $g : X' \to X$ the pull-back of f along $X \to S$. Then $g^*\alpha = \mathcal{N}_f * f^*\alpha$.*

Lemma (1.8). *Let X/S be a separated algebraic space, α a proper relative zero-cycle on X/S and $p : U \to X$ an étale morphism, finite over* $\operatorname{Supp}(\alpha)$*. Then* $\operatorname{Image}(p^*\alpha) = p^{-1}(\operatorname{Image}(\alpha))$*.*

Proof. As the image and composition commutes with étale base change, it is enough to show the equality on an étale cover $S' \to S$. Since the image of α is finite over S, we can thus assume that $p|_{\operatorname{Image}(\alpha)}$ is a trivial étale cover. Then both sides of the equality become disjoint unions of copies of $\operatorname{Image}(\alpha)$. □

Proposition (1.9) (Étale descent)**.** *Let X/S be a separated algebraic space and let $p : U \to X$ be an étale surjective morphism. Let π_1 and π_2 be the projections of $U \times_X U$ onto the two factors. Let β be a proper relative cycle on U/S such that the π_i's are finite over the support of β and $\pi_1^*\beta = \pi_2^*\beta$. Then there is a unique proper relative cycle α on X/S such that $\beta = p^*\alpha$.*

Proof. Let $W \hookrightarrow U$ be the image of β. Then $\pi_1^{-1}(W) = \pi_2^{-1}(W)$ by Lemma (1.8) and thus we obtain by étale descent, a closed subspace $Z \hookrightarrow X$ such that $p^{-1}(Z) = W$. Replacing X with Z we can thus assume that X/S is finite and that p is finite and étale.

As X/S is finite, there is, for any point $s \in S$, an étale neighborhood $S' \to S$ of s such that $U \times_S S' \to X \times_S S'$ has a section s which is an open and closed immersion. We define $\alpha' : S' \to \Gamma^\star(X/S)$ as $s^*(\beta \times_S S')$. As $\pi_1^*\beta = \pi_2^*\beta$, it follows that α' is independent on the choice of section. Furthermore, it follows that the pull-backs of α' along the two projections of $S' \times_S S'$ coincide. By étale descent we obtain, locally around s, a unique family $\alpha : S \to \Gamma^\star(X/S)$ as in the proposition. □

Definition (1.10). Let S be the spectrum of a field k and let α be a relative zero-cycle on X/S and let $x \in X$ be a point. Let $Z = \operatorname{Supp}(\alpha)$. If $x \notin Z$, then we let $\deg_x \alpha = \operatorname{mult}_x \alpha = 0$. If $x \in Z$, then we let $\deg_x \alpha$ be the degree of $\alpha|_U$ for any open neighborhood $U \subseteq X$ such that $U \cap Z = \{x\}$ and we let $\operatorname{mult}_x \alpha$ be the rational number such that $(\operatorname{mult}_x \alpha) \deg\bigl(k(x)/k\bigr) = \deg_x \alpha$. The *geometric multiplicity* of α at x, denoted $\operatorname{geom.mult}_x \alpha$ is the multiple of mult_x and $\deg_{\text{insep}}\bigl(k(x)/k\bigr)$).

Let S be an arbitrary scheme and let α be a proper relative zero-cycle on X/S. Let $x \in X$ be a point with image $s \in S$. Then we let the degree (resp. multiplicity, resp. geometric multiplicity) of α at x, be the degree (resp. multiplicity, resp. geometric multiplicity) of α_s at x. Here α_s denotes the relative zero-cycle $s^*\alpha$ on $X_s/\operatorname{Spec}(k(s))$.

Definition (1.11). Let S be an arbitrary scheme and let α be a relative zero-cycle on X/S. The *underlying cycle* of α is the zero-cycle

$$\operatorname{cycl}(\alpha) = \sum_{x \in \operatorname{Supp}(\alpha)_{\max}} \operatorname{mult}_x(\alpha) \left[\overline{\{x\}}\right].$$

Remark **(1.12).** If α is a relative zero-cycle on $X/\operatorname{Spec}(k)$, then

$$\deg(\operatorname{cycl}(\alpha)) = \sum_{x \in \operatorname{Supp}(\alpha)} \operatorname{mult}_x(\alpha) \deg\bigl(k(x)/k\bigr) = \sum_{x \in \operatorname{Supp}(\alpha)} \deg_x \alpha = \deg(\alpha).$$

The assignment $\alpha \mapsto \operatorname{cycl}(\alpha)$ induces a one-to-one correspondence between relative zero-cycles on X/k and cycles with *quasi-integral* coefficients [Ryd08b, Prop. 9.11].

Proposition (1.13). *Let k be a field, let α be a relative zero-cycle on $X/\operatorname{Spec}(k)$ and let $x \in X$ be a point. Let k'/k be a field extension and α' be the relative zero-cycle on $X_{k'}/k'$ given by pull-back. Then*

(i) *The degree of α at x equals the sum of the degrees of α' at the points above x.*
(ii) *The geometric multiplicities of α at x and of α' at any point x' above x coincide.*
(iii) *Taking the underlying cycle commutes with the base change k'/k, that is,* $\operatorname{cycl}(\alpha)_{k'} = \operatorname{cycl}(\alpha')$.

Let $p : U \to X$ be an étale morphism and $u \in U$ a point mapping to x. Then

(iv) *The multiplicity of α at x and of $p^*\alpha$ at u coincide.*
(v) *Taking the underlying cycle commutes with the pull-back along p, that is,* $p^* \operatorname{cycl}(\alpha) = \operatorname{cycl}(p^*\alpha)$.

Let k/k_0 be a field extension, then

(vi) *The multiplicity of* α *at* x *and the multiplicity of the family* $\mathcal{N}_{k/k_0} * \alpha$ *on* $X/\operatorname{Spec}(k_0)$ *at* x *coincide.*

Proof. Follows easily from Remark (1.12) and the observation that the degree of $\mathcal{N}_{k/k_0} * \alpha$ at x is $\deg(k/k_0)\deg_x \alpha$. $\square$

Definition (1.14) (Trace)**.** Let $f : X \to S$ be an affine morphism and let $\alpha : S \to \Gamma^\star(X/S)$ be a proper relative zero-cycle on X/S. The *trace* of α is the $\mathcal{O}_S$-module homomorphism $\operatorname{tr}(\alpha) : f_*\mathcal{O}_X \to \mathcal{O}_S$ given as the composition of

$$f_*\mathcal{O}_X \to \Gamma^d_{\mathcal{O}_S}(f_*\mathcal{O}_X), \quad g \mapsto \gamma^1(g) \times \gamma^{d-1}(1)$$

and $\alpha^* : \prod_d \Gamma^d_{\mathcal{O}_S}(f_*\mathcal{O}_X) \to \mathcal{O}_S$.

If $Z = \operatorname{Image}(\alpha)$, then the trace of α factors through $f_*\mathcal{O}_X \twoheadrightarrow f_*\mathcal{O}_Z$. If $\mathcal{F}$ is a sheaf on X such that $f_*\mathcal{F}$ is locally free, then $\operatorname{tr}(\mathcal{N}_\mathcal{F})$ is the usual trace of the representation $f_*\mathcal{O}_X \to \operatorname{End}_{\mathcal{O}_S}(f_*\mathcal{F})$.

2. Non-proper relative zero-cycles

We now introduce the notion of *non-proper* relative zero-cycles, or equivalently, non-proper families of zero-cycles, as a first step towards the generalization to higher dimensions. We define non-proper families in great generality, including non-separated schemes and families with support which is not closed.

A non-proper family of zero-cycles should be viewed as an analog of a subscheme $Z \hookrightarrow X \times_S T$ which is flat, locally quasi-finite and locally of finite presentation over T, but not necessarily proper. Every such subscheme Z also defines a non-proper family. More generally, we assign a non-proper family to any coherent sheaf which has finite flat dimension in Section 7. For an étale morphism $p : U \to X$ we define the pull-back p^* of relative zero-cycles which is the ordinary inverse image for flat subschemes.

Definition (2.1). Let X be an algebraic space locally of finite type over S and let $Z \hookrightarrow X$ be a locally closed subset such that $Z \to S$ is locally quasi-finite. A *neighborhood of* X/S *adapted to* Z is a commutative square

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\circ"] & X \arrow[d] \\ T \arrow[r, "g"] & S \end{tikzcd}$$

such that $U \to X \times_S T$ is étale and $p^{-1}(Z) \to T$ is *finite*. We will denote such a neighborhood with (U, T, p, g). If g is étale (resp. smooth) then we say that (U, T, p, g) is an étale (resp. smooth) neighborhood.

A morphism of neighborhoods $(U_1, T_1, p_1, g_1) \to (U_2, T_2, p_2, g_2)$ is a pair of morphisms $p : U_1 \to U_2$ and $g : T_1 \to T_2$ such that

$$\begin{tikzcd} U_1 \arrow[r, "p"'] \arrow[d] \arrow[rr, bend left, "p_1"] & U_2 \arrow[r, "p_2"'] \arrow[d] & X \arrow[d] \\ T_1 \arrow[r, "g"] \arrow[rr, bend right, "g_1"'] & T_2 \arrow[r, "g_2"] & S \end{tikzcd}$$

is commutative.

Remark **(2.2).** If (p, g) is a morphism of neighborhoods as in the definition, then (U_1, T_1, p, g) is a neighborhood of U_2/T_2 adapted to $p_2^{-1}(Z)$. In fact, as $U_1 \to X$ and $U_2 \to X$ are étale it follows that $U_1 \to U_2 \times_{T_2} T_1$ is étale. Moreover, $U_1 \to U_2 \times_{T_2} T_1$ is proper over $p_2^{-1}(Z) \times_{T_2} T_1$.

Recall that a subset $Z \subseteq X$ is *retro-compact* if $Z \cap U$ is quasi-compact for any quasi-compact open subset $U \subseteq X$ [$\mathrm{EGA}_{\mathrm{III}}$, Def. 0.9.1.1]. If X is locally noetherian, then any subset $Z \subseteq X$ is retro-compact.

Definition (2.3). Let X be an algebraic space locally of finite type over S. A (non-proper) relative zero-cycle on X/S consists of the following data

(i) A locally closed retro-compact subset Z of X — the *support* of the cycle.
(ii) For every neighborhood (U, T, p, g) of X/S adapted to Z, a proper family of zero-cycles $\alpha_{U/T} : T \to \Gamma^\star(U/T)$ with support $p^{-1}(Z)$.

satisfying the following conditions:

(a) The support $Z \to S$ is equidimensional of relative dimension zero, i.e., locally quasi-finite and every irreducible component of Z dominates an irreducible component of S.
(b) For every morphism $(p, g) : (U_1, T_1, p_1, g_1) \to (U_2, T_2, p_2, g_2)$ of neighborhoods we have that

$$\alpha_{U_1/T_1} = p'^*(g^* \alpha_{U_2/T_2})$$

where $p' : U_1 \to U_2 \times_{T_2} T_1$ is the canonical étale morphism.

A *non-proper family of zero-cycles* on X/S parameterized by an S-scheme T, is a relative zero-cycle on $X \times_S T/T$.

Remark **(2.4).** If X/S is separated, then every proper relative zero-cycle α on X/S determines a unique non-proper relative zero-cycle with the same support and such that $\alpha_{X/S} = \alpha$. Conversely, a non-proper relative zero-cycle is proper if and only if its support is proper over S. In fact, if Z/S is proper then for every neighborhood (U, T, p, g), we have that $\alpha_{U/T}$ is determined by $\alpha_{X/S}$ according to condition (b).

Definition (2.5). We let $Cycl_0(X/S)$ be the set of relative zero-cycles on X/S. We also let $Cycl^0_{X/S}$ denote the functor from S-schemes to sets such that $Cycl^0_{X/S}(T) = Cycl^0(X \times_S T)$ and the pull-back is the natural map.

As $\Gamma^\star(U/T)$ is representable, it immediately follows that $Cycl^0_{X/S}$ is an fppf-sheaf.

(2.6) *Push-forward* — If $f : X \hookrightarrow Y$ is a quasi-compact immersion, and α is a non-proper relative zero-cycle on X/S, then there is an induced non-proper relative zero-cycle $f_*\alpha$ on Y. More generally, if $f : X \to Y$ is a morphism, and α is a non-proper relative zero-cycle on X/S such that $f(\mathrm{Supp}(\alpha))$ is locally closed and $f|_{\mathrm{Supp}(\alpha)}$ is proper onto its image, then we can define $f_*\alpha$. In particular, this is the case if f is proper and $\mathrm{Supp}(\alpha) \subseteq X$ is closed or if $\mathrm{Supp}(\alpha)/S$ is proper and Y/S is separated.

(2.7) *Addition of cycles* — Let α and β be relative zero-cycles on X/S with supports Z_α and Z_β. If Z_α and Z_β are closed in $Z_\alpha \cup Z_\beta$, e.g., if Z_α and Z_β are closed in X, then there is a relative zero-cycle $\alpha + \beta$ on X/S with support $Z_\alpha \cup Z_\beta$ defined by $(\alpha+\beta)_{U/T} = \alpha_{U/T} + \beta_{U/T}$ for any neighborhood (U,T) adapted to $Z_\alpha \cup Z_\beta$.

The condition on Z_α and Z_β is equivalent with the condition that $f : Z_\alpha \amalg Z_\beta \to Z_\alpha \cup Z_\beta$ is proper. This is necessary to ensure that a neighborhood adapted to $Z_\alpha \cup Z_\beta$ also is adapted to Z_α and Z_β. This condition also implies that $Z_\alpha \cup Z_\beta$ is locally closed. We have that $\alpha + \beta = f_*(\alpha \amalg \beta)$.

(2.8) *Flat zero-cycles* — If Z/S is locally quasi-finite, flat and locally of finite presentation, then there is a non-proper family $\mathcal{N}_{Z/S}$ of zero-cycles on Z/S with support Z_{red}. This family is defined by $(\mathcal{N}_{Z/S})_{U/T} = \mathcal{N}_{p^{-1}(Z)/T}$ on any projection (U,T,p). The compatibility condition (b) follows from the functoriality of the norm.

(2.9) *Relative cycles on smooth curves* — If X/S is a smooth curve, then any relative cycle α on X/S is the norm family $\mathcal{N}_{Z/S}$ of a unique subscheme $Z \hookrightarrow X$.

(2.10) *Pull-back* — If α is a relative zero-cycle on X/S and $f : U \to X$ is an étale morphism, then we define the relative zero-cycle $f^*\alpha$ on U/S as follows. The support of $f^*\alpha$ is $f^{-1}(\mathrm{Supp}(\alpha))$ and for any neighborhood (V,T,p,g) adapted to $f^{-1}(\mathrm{Supp}(\alpha))$ we let $(f^*\alpha)_{V/T} = \alpha_{V/T, f\circ p, g}$.

Lemma (2.11) (Existence of étale neighborhoods)**.** *Let X/S be an algebraic space, locally of finite type. Let $Z \hookrightarrow X$ be a locally closed subspace such that $Z \to S$ is locally quasi-finite. Then for every point $z \in Z$, there exists an étale neighborhood (U,T,p,g) of X/S adapted to Z such that there exists $u \in U$ such that $z = p(u)$ and $k(z) \to k(u)$ is an isomorphism. Furthermore, we can assume that u is the only point in its fiber $U_t \cap p^{-1}(Z)$. If X/S is a scheme or a separated algebraic space, then we can furthermore choose $U \to X \times_S T$ as an open immersion.*

Proof. Replacing X with an étale neighborhood of z in X, we can assume that X is a scheme [Knu71, Thm. II.6.4] and that z is the only point in its fiber over S. It then follows from [$\mathrm{EGA_{IV}}$, Thm. 18.12.1 and Rmk. 18.12.2] that there is an étale morphism $g : T \to S$, a point $z' \in Z_T = Z \times_S T$ above z such that $k(z) \cong k(z')$ and an open neighborhood V of z' in Z_T such that

$V \to T$ is finite. Any open subscheme U of $X \times_S T$ such that $U \cap Z_T = V$ gives an étale neighborhood as in the lemma.

If X/S is a scheme, then the last statement follows immediately, as we can skip the first step in the construction of U. If X/S is a separated algebraic space then nevertheless Z/S is a scheme [LMB00, Thm. A.2] and the statement follows. □

Remark **(2.12).** If X/S is a *locally separated* algebraic space, then we can choose $U \to X \times_S T$ as an open immersion if we drop the condition that $k(u)/k(z)$ is a trivial extension. This follows from the fact that if S is a *strictly* henselian local scheme and if $Z \to S$ is a locally separated quasi-finite morphism, then $Z \to S$ is finite over an open subset containing the closed fiber. This can be shown similarly as [LMB00, Lem. A.1].

Proposition (2.13). *The support of a relative zero-cycle is universally open and hence universally equidimensional of relative dimension zero.*

Proof. This follows immediately from Lemma (2.11) as the support of a proper relative zero-cycle is universally open. □

Definition (2.14). Let X/S be locally of finite type and let α be a relative zero-cycle on X/S. Let $x \in X$. The degree (resp. multiplicity, resp. geometric multiplicity) of α at x is the corresponding number of $(\alpha_s)|_U$ at x for any neighborhood $U \subseteq X_s$ of x such that $\operatorname{Supp}(\alpha_s)|_U$ is finite. We say that α is *non-degenerate* or *étale* at x if $\operatorname{geom.mult}_x(\alpha) = 1$.

Proposition (2.15). *Let X/S be an algebraic space, locally of finite type, and let α be a relative zero-cycle on X/S. Then the function* $\operatorname{geom.mult} : \operatorname{Supp}(\alpha) \to \mathbb{N}$, $x \mapsto \operatorname{geom.mult}_x(\alpha)$ *is upper semi-continuous. In particular, α is étale at an open subset of* $\operatorname{Supp}(\alpha)$.

Proof. Let $x \in \operatorname{Supp}(\alpha)$ be a point with geometric multiplicity m. We have to show that the geometric multiplicity is at most m in a neighborhood of x. This can be checked on any étale neighborhood and we can thus assume that $\operatorname{Supp}(\alpha) \to S$ is finite, that x is the only point in its fiber $\operatorname{Supp}(\alpha)_s$ and that $k(x)/k(s)$ is purely inseparable. Then m is the degree of α at s. As the degree of α is m in a neighborhood of S, the geometric multiplicity of α is at most m in a neighborhood of x. □

Definition (2.16). Let $X/\operatorname{Spec}(k)$ be locally of finite type and let α be a relative zero-cycle on $X/\operatorname{Spec}(k)$. If $\operatorname{Supp}(\alpha)$ is finite, then α is a proper relative zero-cycle and $\deg(\alpha)$ is defined. If $\operatorname{Supp}(\alpha)$ is infinite, then we let $\deg(\alpha) = \infty$.

Proposition (2.17). *Let X/S be a* separated *algebraic space, locally of finite type, and let α be a relative zero-cycle on X/S. Then the function* $\deg : S \to \mathbb{N} \cup \{\infty\}$, $s \mapsto \deg(\alpha_s)$ *is lower semi-continuous, i.e., for every $d \in \mathbb{N}$, the subset of S where* $\deg$ *is at most d is closed. The relative cycle α is* proper *if and only if* $\deg$ *is finite and locally constant.*

Proof. Let $s \in S$ such that $d = \deg(\alpha_s)$ is finite. Let $Z = \operatorname{Supp}(\alpha)$. Let $(S', s') \to (S, s)$ be an étale neighborhood such that $Z \times_S S' = Z'_1 \amalg Z'_2$ where $Z'_1 \to S'$ is finite of rank d and $(Z'_2)_{s'}$ is empty [$\mathrm{EGA}_{\mathrm{IV}}$, Thm. 18.12.1]. Then

$\deg(\alpha \times_S S') \geq d$ over the image of $Z_1' \to S'$ which is an open neighborhood of s' as $Z \to S$ is universally open. Hence deg is lower semi-continuous.

Assume that $\deg(s) = d$ for all $s \in S$. Then in the above construction it follows that the images of Z_1' and Z_2' does not intersect. It follows that over the image of Z_1', which is open, $\mathrm{Supp}(\alpha \times_S S')$ is finite. By étale descent, so is $\mathrm{Supp}(\alpha)$ in a neighborhood of s. □

Proposition (2.18) (Étale descent). *Let X/S be an algebraic space and let $p : U \to X$ be an étale morphism. Let π_1 and π_2 be the projections of $U \times_X U$ onto the first and second factors. Let β be a relative zero-cycle on U/S such that $\pi_1^*\beta = \pi_2^*\beta$. Then there is a unique relative zero-cycle α on X/S with support contained in $p(U)$ such that $\beta = p^*\alpha$.*

Proof. As $\pi_1^{-1}(\mathrm{Supp}(\beta)) = \pi_2^{-1}(\mathrm{Supp}(\beta))$ we obtain by étale descent of quasi-compact immersions [SGA$_1$, 5.5 and 7.9], a locally closed retro-compact subscheme $Z \hookrightarrow X$ such that $p^{-1}(Z) = \mathrm{Supp}(\beta)$ and Z is contained in $p(U)$. The support of α will be Z.

Let (V, T, q, g) be a neighborhood of X/S adapted to Z. We will construct a canonical proper family on V/T which is compatible with β. We let $W = U \times_X V$ such that

$$\begin{tikzcd} W \times_V W \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & W \arrow[r] \arrow[d, "r"] & V \arrow[d, "q'"] \\ U_T \times_{X_T} U_T \arrow[r, shift left] \arrow[r, shift right] & U_T \arrow[r, "p"] & X_T \end{tikzcd}$$

is cartesian. The family $r^*(\beta \times_S T)$ is compatible with respect to the two projections of $W \times_V W$. Replacing S, X and U with T, V and W respectively, we can thus assume that X/S itself is adapted to Z.

The support of β is $p^{-1}(Z)$. Lemma (2.11) gives an étale neighborhood (V, T, q, g) of U/S adapted to $p^{-1}(Z)$ such that $p^{-1}(Z)$ is contained in the image of $q : V \to U$. If we construct a unique proper family $\alpha' : T \to \Gamma^\star(X/S)$ then the existence of the proper family $\alpha : S \to \Gamma^\star(X/S)$ follows by étale descent. We can thus replace S with T and assume that there is an étale neighborhood (V, S, q, g) of U/S adapted to $p^{-1}(Z)$. By the compatibility of the family β, we can finally replace U with V. Then β is proper and the result follows from Proposition (1.9). □

Remark **(2.19).** An easy special case of the proposition is the following situation. Let X/S be an algebraic space and let $X = \bigcup_i U_i$ be an open covering. Given non-proper families α_i on U_i/S which coincide on the intersections, there is then a unique family α on U/S such that $\alpha_{U_i/S} = \alpha_i$.

Corollary (2.20). *In the definition of non-proper relative zero-cycles, it is enough to only consider étale neighborhoods (U, T, p, g) of X/S, i.e., neighborhoods such that $g : T \to S$ is étale. Furthermore, we can require that U and T are affine.*

Proof. Follows immediately from Lemma (2.11) and Proposition (2.18). □

(2.21) *Composition of relative zero-cycles* — Let X/Y and Y/S be algebraic spaces locally of finite type. Let α be a relative zero-cycle on Y/S

and β a relative zero-cycle on $f : X \to Y$. Then there is a natural relative zero-cycle $\alpha * \beta$ on X/S with support $Z = f^{-1}\mathrm{Supp}(\alpha) \cap \mathrm{Supp}(\beta)$ such that when $f : X \to Y$ is étale, we have that $f^*\alpha = \alpha * \mathcal{N}_f$. Also $*$ will be associative. We define $\alpha * \beta$ as follows.

It is by Proposition (2.18) enough to define $\alpha * \beta$ on an étale cover of X. Replacing X and Y with étale covers, we can thus assume that X and Y are separated. Let $Z = \mathrm{Supp}(\alpha * \beta)$ and let (U, T, p, g) be a neighborhood of X/S adapted to Z. Let $p' : U \to X_T$ be the induced morphism. Let $W = f_T(p'(p^{-1}(Z))) \subseteq \mathrm{Supp}(\alpha) \times_S T$ which is an open subset as $\mathrm{Supp}(\beta) \to Y$ is universally open. Choose an open subset $V \subseteq Y_T$ restricting to W and let $U' = p'^{-1}(f_T^{-1}(V)) \subseteq U$. Then $p^{-1}(Z) \subseteq U'$ and it is enough to define $(\alpha * \beta)_{U'/T}$.

As $p^{-1}(Z) \to W$ is surjective, we have that $W \to T$ is proper and thus $\alpha_{V/T}$ is defined. As $p^{-1}(Z) \to W$ is proper $\alpha_{U'/\mathrm{Image}(\alpha_{V/T})}$ is also defined. We let $(\alpha * \beta)_{U'/T} = \alpha_{V/T} * \alpha_{U'/\mathrm{Image}(\alpha_{V/T})}$.

Proposition (2.22). *Let X/S be an algebraic space locally of finite type and let α be a relative zero-cycle on X/S. Let $f : S' \to S$ be an étale morphism and denote by $g : X' \to X$ the pull-back of f along $X \to S$. Then $g^*\alpha = \mathcal{N}_f * f^*\alpha$.*

Proof. This follows from the construction of $*$ for non-proper relative cycles and the proper case, Proposition (1.7). □

The compatibility condition (b) of Definition (2.3) implies the following compatibility.

Corollary (2.23). *Let $X \to S$ be an algebraic space, locally of finite type which factors through an étale morphism $h : S' \to S$. If α is a relative zero-cycle on X/S then $\mathcal{N}_{S'/S} * \alpha_{X/S'} = \alpha_{X/S}$. In particular, there is a one-to-one correspondence between relative zero-cycles on X/S' and relative zero-cycles on X/S.*

Proof. Let $p : X' \to X$ be the pull-back of $h : S' \to S$. The factorization $X \to S' \to S$ induces an open section $s : X \to X'$ of p. Then

$$\begin{aligned}\alpha_{X/S} &= s^*p^*(\alpha_{X/S}) = s^*(\mathcal{N}_{S'/S} * h^*(\alpha_{X/S})) \\ &= s^*(\mathcal{N}_{S'/S} * \alpha_{X'/S'}) = \mathcal{N}_{S'/S} * s^*\alpha_{X'/S'} = \mathcal{N}_{S'/S} * \alpha_{X/S'}\end{aligned}$$

by Proposition (2.22). □

Proposition (2.24). *Let α be a relative zero-cycle on X/S with support $Z \hookrightarrow X$. There is then a unique locally closed subspace* $\mathrm{Image}(\alpha) \hookrightarrow X$ *such that for any neighborhood (U', S', p, g) we have that* $\mathrm{Image}(\alpha_{U'/S'}) \subseteq p^{-1}(\mathrm{Image}(\alpha))$ *with equality if g is smooth. Moreover,* $\mathrm{Image}(\alpha)_{\mathrm{red}} = Z$.

Proof. Let $z \in Z$ and let (U', S', p, g) be a smooth neighborhood such that z is in the image of $p(U')$. Such a neighborhood exists by Lemma (2.11). Let $S'' = S' \times_S S'$, $X' = X \times_S S'$, $X'' = X \times_S S''$ and let π_1, π_2 be the two projections $X'' = X' \times_X X' \to X'$. Let $U''_i = \pi_i^* U'$, $i = 1, 2$ and $U'' = U' \times_X U' = U''_1 \times_{X''} U''_2$.

The image $W' = \mathrm{Image}(\alpha_{U'/S'})$ is an infinitesimal neighborhood of $Z' = p^{-1}(Z)$. As the image of a proper family of zero-cycles commutes with smooth base change it follows that

$$W_i'' = \pi_i^{-1}(W') = \mathrm{Image}(\alpha_{U_i''/S''}).$$

Let $W'' = \mathrm{Image}(\alpha_{U''/S''})$. By the compatibility of α we have that $\alpha_{U''/S''} = \pi_i^* \alpha_{U_i''/S''}$. Furthermore, as U''/U_i'' is étale we have by Lemma (1.8) that the inverse image of W_i'' along $U'' \to U_i''$ is W''.

Thus $W' \hookrightarrow U'$ is a closed subscheme with support Z' such that the inverse images of W' along the projections of $U'' \to U' \times_X U'$ coincide. By fppf descent it thus follows that there is a closed subscheme $W \hookrightarrow p(U')$ such that $W' = p^{-1}(W)$. In particular, we have that $W_{\mathrm{red}} = Z \cap p(U')$. As it is obvious that W does not depend on the choice of smooth neighborhood, there is a unique locally closed subspace $\mathrm{Image}(\alpha)$ such that $p^{-1}(\mathrm{Image}(\alpha)) = W$. □

If α is a relative zero-cycle on X/S with image Z, then α is the push-forward of a relative zero-cycle on Z/S along the immersion $Z \hookrightarrow X$. Also note that if α is étale with image Z, then Z/S is étale and $\alpha = \mathcal{N}_{Z/S}$.

(2.25) *Trace* — Let α be a relative zero-cycle on X/S. Let Z be the *image* of α. For every neighborhood (U, T, p, g) we obtain a *trace map* $h_* \mathcal{O}_{p^{-1}(Z)} \to \mathcal{O}_T$, cf. Definition (1.14). Here h denotes the morphism $p^{-1}(Z) \to U \to T$.

(2.26) *Fundamental class* — Let S be locally noetherian and let X/S be separated and locally of finite type. Let α be a relative zero-cycle on X/S and $Z \hookrightarrow X$ its image. Let (U, T, p, g) be an étale neighborhood. By duality, cf. Appendix A, the trace map corresponds to a class in

$$\mathrm{H}^0(p^{-1}(Z), h^! \mathcal{O}_T) = \mathrm{H}^0(p^{-1}(Z), p^* \mathcal{D}^\bullet_{Z/S})$$

By the compatibility condition on α, it follows that that this class is the restriction of a unique class in $\mathrm{H}^0(Z, \mathcal{D}^\bullet_{Z/S})$, the *relative fundamental class* of α, cf. [AEZ78, Prop. II.2].

Let $j : Z \hookrightarrow X$ be the inclusion and assume that j is closed. By duality, we then also have that

$$\mathrm{H}^0(Z, \mathcal{D}^\bullet_{Z/S}) = \mathrm{H}^0(Z, j^! \mathcal{D}^\bullet_{X/S}) = \mathrm{Ext}^0_X(j_* \mathcal{O}_Z, \mathcal{D}^\bullet_{X/S}).$$

This gives a unique class in $\mathrm{H}^0_{|Z|}(X, \mathcal{D}^\bullet_{X/S})$. In particular, if X/S is smooth of relative dimension n, then this is a class in $\mathrm{H}^n_{|Z|}(X, \Omega^n_{X/S})$.

When S is of characteristic zero, or the characteristic of $k(z)$ exceeds the geometric multiplicity of α at z for every $z \in Z$, then the relative fundamental class, in either $\mathrm{H}^0(Z, \mathcal{D}^\bullet_{Z/S})$ or $\mathrm{H}^0_{|Z|}(X, \mathcal{D}^\bullet_{X/S})$, uniquely determines α.

(2.27) *Fundamental class II* — Let S be locally noetherian, let $q : B \to S$ be smooth of relative dimension r and let $f : X \to B$ be separated and locally of finite type. Let α be a relative zero-cycle on X/B. Then we have the relative fundamental class $c_\alpha \in \mathrm{H}^0(Z, \mathcal{D}^\bullet_{Z/B})$. This gives an element

$$\begin{aligned}
\mathrm{Ext}^0_Z\big(h^*(\Omega^r_{B/S}), h^! \mathcal{O}_B \otimes_{\mathcal{O}_Z} h^*(\Omega^r_{B/S})\big) &= \mathrm{Ext}^0_Z\big(h^*(\Omega^r_{B/S}), h^!(\Omega^r_{B/S})\big) \\
&= \mathrm{Ext}^{-r}_Z\big(h^*(\Omega^r_{B/S}), h^! \mathcal{D}^\bullet_{B/S}\big) \\
&= \mathrm{Ext}^{-r}_Z\big(h^*(\Omega^r_{B/S}), \mathcal{D}^\bullet_{Z/S}\big).
\end{aligned}$$

If α is induced from a relative cycle of dimension r on X/S, then this class is induced by a class in $\operatorname{Ext}_Z^{-r}\big(\Omega^r_{Z/S}, \mathcal{D}^\bullet_{Z/S}\big)$ as will be shown in Theorem (16.1).

(2.28) *Interpretation with multiplicative laws* — If $h : U = \operatorname{Spec}(B) \to T = \operatorname{Spec}(A)$ is a morphism of affine schemes, then a morphism $\alpha_{U/T} : T \to \Gamma^\star(U/T)$ corresponds to a multiplicative A-law $B \to A$ [Ryd08b, Thm. 2.3]. Such a law corresponds to multiplicative *maps* $h'_*\mathcal{O}_{U\times_T T'} \to \mathcal{O}_{T'}$ for every *smooth* T-scheme T' (it is enough to take $T' = \mathbb{A}^n_T$) such that for any morphism $\varphi : T'_1 \to T'_2$ the following diagram commutes

$$\begin{tikzcd}
(h'_1)_*\mathcal{O}_{U\times_T T'_1} \arrow[r] \arrow[d] & (h'_1)_*\varphi^*\mathcal{O}_{U\times_T T'_2} \arrow[d] \\
\mathcal{O}_{T'_1} \arrow[r] & \varphi^*\mathcal{O}_{T'_2}.
\end{tikzcd}$$

In the definition of a relative zero-cycle, we can thus instead of giving a proper zero-cycle $\alpha_{U/T}$ on every neighborhood (U, T) instead give a multiplicative map $h_*\mathcal{O}_U \to \mathcal{O}_T$ with support on $p^{-1}(Z)$ such that these maps satisfy a similar compatibility condition.

3. Condition (T)

In this section we give a topological condition on a morphism closely related to conditions such as equidimensional and universally open.

Definition (3.1). Let $f : X \to S$ be a morphism. An irreducible component $X_i \hookrightarrow X$ is *dominating* over S if $\overline{f(X_i)}$ is an irreducible component of S. We let $X_{\mathrm{dom}/S} \subseteq X$ be the union of the irreducible components which are dominating over S. If $X = X_{\mathrm{dom}/S}$ then we say that f is *componentwise dominating.*

Remark **(3.2).** Let $X \to S$ be a morphism. If S has a finite number of irreducible components with generic points $\xi_1, \xi_2, \dots, \xi_n$, then $X_{\mathrm{dom}/S}$ is the underlying set of the schematic closure of $X \times_S \coprod_i \operatorname{Spec}(\mathcal{O}_{S,\xi_i})$ in X. If $X \to S$ is open, then $X_{\mathrm{dom}/S} = X$.

Definition (3.3). Let $f : X \to S$ be a morphism locally of finite type. We let $X_{\dim_S = r}$ (resp. $X_{\dim_S > r}$) be the subset of X consisting of points $x \in X$ with $\dim_x\big(X_{f(x)}\big) = r$ (resp. $> r$). By Chevalley's theorem [EGA$_{\mathrm{IV}}$, Thm. 13.1.3], this is a locally closed (resp. closed) subset.

Let $f : X \to S$ be locally of finite type. Recall [EGA$_{\mathrm{IV}}$, 13.3, Err$_{\mathrm{IV}}$, 35] that f

(i) is *equidimensional* if f is componentwise dominating, and locally on S there exists an integer r such that the fibers of f are equidimensional of dimension r,
(ii) is *equidimensional at* $x \in X$ if $f|_U$ is equidimensional for some open neighborhood U of x,
(iii) is *locally equidimensional* if f is equidimensional at every point $x \in X$.

Proposition (3.4). *Let $f : X \to S$ be a morphism locally of finite type. The following conditions are equivalent:*

(i) *For every integer r, the subscheme $X_{\dim_S = r}$ is equidimensional of dimension r over S.*
(ii) *For every integer r, every irreducible component of $X_{\dim_S = r}$ dominates an irreducible component of S.*
(iii) *Every point $x \in X$ is contained in an irreducible component W of X which is equidimensional over S at x.*
(iv) *Every point $x \in X$ which is generic in its fiber $X_{f(x)}$ is contained in an irreducible component W of X which is equidimensional over S at x.*

Moreover, these conditions are satisfied if f is universally open or if the irreducible components of X are equidimensional over S, e.g., if f is locally equidimensional.

Proof. By definition, (i) is equivalent to (ii) and trivially (iii) implies (iv). It is obvious that (i) implies (iii). If (iv) is satisfied, then any irreducible component of $X_{\dim_S = r}$ is contained in, and hence equal to, an irreducible component which is equidimensional of dimension r. This shows that (iv) implies (i).

If f is universally open, then (iv) is satisfied by [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 14.3.13]. If f is locally equidimensional, then (i) is satisfied. □

Definition (3.5). We say that X/S satisfies condition (T) when the equivalent conditions of Proposition (3.4) are satisfied. We say that X/S satisfies (T) universally if $X \times_S S'/S'$ satisfies (T) for any base change $S' \to S$.

Note that if X/S satisfies (T), then X'/S' satisfies (T) for any *flat* base change $S' \to S$, cf. [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 13.3.8].

Proposition (3.6). *Let $f : X \to S$ be locally of finite type. The following are equivalent.*

(i) *f satisfies* (T) *universally.*
(ii) *$f' : X' \to S'$ is componentwise dominating for every morphism $S' \to S$.*
(iii) *$f' : X' \to S'$ is componentwise dominating for every morphism $S' \to S$ where S' is the spectrum of a valuation ring.*

If S is locally noetherian, then these statements are equivalent with:

(iv) *$f' : X' \to S'$ is componentwise dominating for every morphism $S' \to S$ where S' is the spectrum of a discrete valuation ring.*
(v) *f is universally open.*

Proof. Is is clear that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv). If S' is the spectrum of a valuation ring, then f' satisfies condition (T) if and only if f' is componentwise dominating by [$\mathrm{EGA}_{\mathrm{IV}}$, Lem. 14.3.10]. An easy argument then shows that (iii) implies (i). That (iv) implies (v) is [$\mathrm{EGA}_{\mathrm{IV}}$, Cor. 14.3.7] and finally (v) implies (i) by [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 14.3.13]. □

4. Families of higher-dimensional cycles

In this section, we define higher-dimensional relative cycles. The support of a cycle will be universally open, Proposition (4.7), but a priori, the support only satisfies the weaker condition (T) of the previous section. We do not require that the support of a relative cycle is equidimensional, nor that its irreducible components are equidimensional. In the sequel, we will often use the following two results.

(i) If $B \to S$ is a smooth morphism, then for every $b \in B$, there is an open neighborhood $U \ni b$ and an étale morphism $U \to \mathbb{A}^r_S$ [$\mathrm{EGA_{IV}}$, Cor. 17.11.4].
(ii) If $Z \to B$ is open (or equidimensional), $B \to S$ is smooth and the composition is flat and locally of finite presentation with Cohen-Macaulay fibers (e.g. smooth), then $Z \to B$ is flat [$\mathrm{EGA_{IV}}$, Thm. 11.3.10, Prop. 15.4.2].

As in previous sections, we work with algebraic spaces X/S locally of finite type. It may appear more natural to assume that X/S is locally of finite presentation, and indeed this is required in several statements. However, even if X/S is of finite presentation, the support, image, representing scheme, etc., of a relative cycle is a subscheme of X which need not be of finite presentation. Of course, it is expected that any relative cycle α on X/S is of finite presentation, i.e., that there exists X_0/S_0 of finite presentation and a relative cycle α_0 on X_0/S_0 which pull-backs to α. If this is the case, then locally there are infinitesimal neighborhoods of the support, image, representing scheme, etc., which are finitely presented. Unfortunately, these neighborhoods are not canonical and do not glue.

Definition (4.1). Let X be an algebraic space, locally of finite type over S and let $Z \hookrightarrow X$ be a locally closed subset. A *projection of X/S adapted to Z* (resp. quasi-adapted to Z) is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d, "\varphi"] & X \arrow[dd] \\ B \arrow[d] & \\ T \arrow[r, "g"] \arrow[ur, phantom, "\circ"] & S \end{tikzcd}$$

such that $U \to X \times_S T$ is étale, $p^{-1}(Z) \to B$ is finite (resp. quasi-finite) and $B \to T$ is smooth. We will denote such a projection with (U, B, T, p, g, φ). If g is étale (resp. smooth) then we say that (U, B, T, p, g, φ) is an étale (resp. smooth) projection.

A morphism of projections $(U_1, B_1, T_1, p_1, g_1, \varphi_1) \to (U_2, B_2, T_2, p_2, g_2, \varphi_2)$ is a triple of morphisms $p : U_1 \to U_2$, $q : B_1 \to B_2$ and $g : T_1 \to T_2$ such that

$$\begin{tikzcd}
U_1 \arrow[r, "p"'] \arrow[rr, bend left, "p_1"] \arrow[d, "\varphi_1"] & U_2 \arrow[r, "p_2"'] \arrow[d, "\varphi_2"] & X \arrow[dd] \\
B_1 \arrow[r, "q"] \arrow[d] & B_2 \arrow[d] & \\
T_1 \arrow[r, "g"] \arrow[rr, bend right, "g_1"'] & T_2 \arrow[r, "g_2"] & S
\end{tikzcd}$$

is commutative.

Definition (4.2). Let X/S be an algebraic space, locally of finite type over S. A *relative cycle* α on X/S consists of the following:

(i) A locally closed retro-compact subset Z of X — the *support* of α.

(ii) For every projection (U, B, T, p, g) of X/S adapted to Z, a proper family of zero-cycles $\alpha_{U/B/T} : B \to \Gamma^\star(U/B)$ with support $p^{-1}(Z)_{\mathrm{dom}/B}$.

satisfying the following conditions:

(a) The support Z satisfies (T).

(b) For every morphism $(p, q, g) : (U_1, B_1, T_1, p_1, g_1) \to (U_2, B_2, T_2, p_2, g_2)$ of projections, we have that

$$\mathcal{N}_{B_1/g^*B_2} * \alpha_{U_1/B_1/T_1} = g^*\alpha_{U_2/B_2/T_2} * \mathcal{N}_{U_1/g^*U_2}.$$

A relative cycle is locally equidimensional (resp. equidimensional of dimension r) if Z/S is locally equidimensional (resp. equidimensional of relative dimension r).

Let us show that condition (b) makes sense. First note that $U_1 \to g^*U_2$ is étale and thus the right-hand side is defined. To define the left-hand side, we can replace the B_i's with the respective images of the universally open morphisms $p_i^{-1}(Z)_{\mathrm{dom}/B_i} \to B_i$. Then as $U_1 \to g^*U_2$ is universally open, it follows that $B_1 \to B_2$ is universally open. Thus $B_1 \to g^*B_2$ is *flat* and of finite presentation [$\mathrm{EGA}_{\mathrm{IV}}$, Thm. 11.3.10, Prop. 15.4.2]. As $B_1 \to g^*B_2$ is quasi-finite $\mathcal{N}_{B_1/g^*B_2}$ is thus defined.

Remark **(4.3).** As $B \to T$ is smooth, there is an open and closed partition of B such that $B \to T$ is equidimensional. It is thus clear that in Definition (4.2), we can assume that $B \to T$ is equidimensional and that $\mathrm{Supp}(\alpha_{U/B/T}) \to B$ is surjective. If the support Z is equidimensional of dimension r, then it is enough to consider projections with B/T smooth of dimension r.

Remark **(4.4).** It is easily seen if α is a relative cycle on X/S, then for any projection (U, B, T, p, g) *quasi-adapted* to $\mathrm{Supp}(\alpha)$ there is a unique *non-proper* relative zero-cycle $\alpha_{U/B/T}$ with support $p^{-1}(Z)_{\mathrm{dom}/B}$. The compatibility condition (b) is then also satisfied for morphisms of quasi-adapted projections.

Proposition (4.5). *Let α be a relative cycle on X/S with support Z. Let $Z_r = Z_{\dim_S = r}$. If (U, B, T, p, g) is a projection such that B/T has relative dimension r, then* $\mathrm{Supp}(\alpha_{U/B/T}) \subseteq p^{-1}(Z_r)$ *with equality if $p^{-1}(Z_r) \to T$ is componentwise dominating. The collection of $\alpha_{U/B/T}$ for which B/T has dimension r, determines a unique equidimensional relative cycle α_r with support Z_r.*

Proof. Let (U, B, T, p, g) be a neighborhood adapted to Z. As $p^{-1}(Z)$ is finite over B, it follows that every point of $p^{-1}(Z)$ has dimension at most r relative to T and that $\mathrm{Supp}(\alpha_{U/B/T}) = p^{-1}(Z)_{\mathrm{dom}/B} \subseteq p^{-1}(Z)_{\dim_T = r} = p^{-1}(Z_r)$.

Let (U, B, T, p, g) be a neighborhood adapted to Z_r and let $p' : U' \to X$ be the restriction of p to the open subset $X \setminus \overline{Z_{\dim_S > r}}$. Then (U', B, T, p', g) is a neighborhood adapted to Z and $\alpha_{U'/B/T}$ determines $(\alpha_r)_{U/B/T}$ uniquely. $\square$

Lemma (4.6) (Existence of étale projections)**.** *Let $f : X \to S$ be an algebraic space, locally of finite type, and α a relative cycle on X/S with support Z. Then for any point $z \in Z$ there is an étale projection (U, B, S, p, g) adapted to Z and $u \in p^{-1}(z)$ such that $u \in p^{-1}(Z)|_{\mathrm{dom}/B}$.*

Proof. Replacing X with an étale cover, we can assume that X is a scheme. Let $r = \dim_z\bigl(X_{f(z)}\bigr)$. There is an open neighborhood $U \subseteq X$ of z and a morphism $U \cap Z_r \to \mathbb{A}^r_S$ which is equidimensional of dimension zero [$\mathrm{EGA_{IV}}$, Prop. 13.3.1 b]. After shrinking U, we can assume that we have a morphism $U \to \mathbb{A}^r_S$ such that $(U \cap Z)_{\mathrm{dom}/\mathbb{A}^r_S} = (U \cap Z_r)$ in a neighborhood of z. The result then follows from Lemma (2.11). $\square$

The following proposition shows that the support of a relative cycle behaves similarly as the support of a flat and finitely presented sheaf. One difference though is that the irreducible components of the support of a flat sheaf always are equidimensional [$\mathrm{EGA_{IV}}$, Prop. 12.1.1.5].

Proposition (4.7). *Let X/S be locally of finite type. The support of a relative cycle α on X/S is universally open. In particular, an equidimensional relative cycle is universally equidimensional and equality always holds in Proposition* (4.5)*.*

Proof. Let α be a relative cycle. It is enough to show that the support Z_r of α_r is universally open over S for every r. This follows from Lemma (4.6) and Proposition (2.13). $\square$

Remark **(4.8).** The support of a single irreducible component of α need not be universally open. For example, if S consists of two secant lines and $X = S$, then there is a relative zero-cycle on X/S with support X but the inclusion of one of the lines is not open. This is also illustrated in the following example.

Example (4.9) ([$\mathrm{EGA_{IV}}$, Rem. 14.4.10 (ii)])**.** Let S be a regular quasi-projective surface and choose a closed point $s \in S$. Let Z_1 be the blow-up of S in s and let $Z_2 = \mathbb{P}^1_S$. Then $(Z_1)_s \cong (Z_2)_s \cong \mathbb{P}^1_s$. We let $Z = Z_1 \amalg_{\mathbb{P}^1_s} Z_2$

be the gluing of Z_1 and Z_2 along the common fiber. This is a scheme [Fer03, Thm. 5.4] with irreducible components Z_1 and Z_2.

Note that $Z_1 \to S$ does not satisfy (T) but that $Z \to S$ satisfies (T). It follows from Chevalley's theorem [EGA$_{\mathrm{IV}}$, Thm. 14.4.1] that $Z \to S$ is universally open but that $Z_1 \to S$ is not universally open. Later on, in Theorem (10.1), we will see that Z/S determines a unique relative cycle on Z/S with underlying cycle $[Z] = [Z_1] + [Z_2]$. Thus, this an example of a relative cycle for which the irreducible components are not equidimensional. This is a phenomenon which does not occur in flat families.

Proposition (4.10) (Étale descent)**.** *Let X/S be locally of finite type and let $p : U \to X$ be an étale morphism. Let π_1 and π_2 be the projections of $U \times_X U$ onto the first and second factors. Let β be a relative cycle on U/S such that $\pi_1^*\beta = \pi_2^*\beta$. Then there is a unique relative cycle α on X/S with support contained in $p(U)$ such that $\beta = p^*\alpha$.*

Proof. Let $W \subseteq U$ be the support of β. Then $\pi_1^{-1}(W) = \pi_2^{-1}(W)$ and by étale descent, we obtain a locally closed retro-compact subset $Z \subseteq p(U)$. If (V, B, T) is a projection adapted to Z/S, then $(U \times_X V, B, T)$ is a projection quasi-adapted to W/S. The relative zero-cycle $\alpha_{U\times_X V/B/T}$ then descends uniquely to a relative zero-cycle $\alpha_{V/B/T}$ by Proposition (2.18). $\square$

Proposition (4.11). *Let α be a relative cycle on X/S. Let (U, B, T, p, g) be a projection such that $B \to T$ factors through an étale morphism $h : T' \to T$. Then $(U, B, T', p, g \circ h)$ is a projection and $\alpha_{U/B/T'} = \alpha_{U/B/T}$.*

Proof. As $T' \to T$ is étale, $U \to X \times_S T$ is étale and $B \to T$ is smooth, it follows that $U \to X \times_S T'$ is étale and that $B \to T'$ is smooth. Thus $(U, B, T', p, g \circ h)$ is a projection. We also have a natural map of projections $(\mathrm{id}_U, \mathrm{id}_B, h) : (U/B/T') \to (U/B/T)$. The compatibility condition for this map is that

$$\mathcal{N}_{B/B\times_T T'} * \alpha_{U/B/T'} = \pi_2^*\alpha_{U/B/T} * \mathcal{N}_{U/U\times_T T'}$$

where the maps are given by the diagram

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_T T' \arrow[r] \arrow[d] & U \arrow[d] \\
B \arrow[r, hook] \arrow[dr, "="'] & B \times_T T' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d] \\
& T' \arrow[r, "h"] & T
\end{tikzcd}$$

But as $B \to B\times_T T'$ is an open immersion, it is obvious that this is equivalent to $\alpha_{U/B/T'} = \alpha_{U/B/T}$. $\square$

Corollary (4.12). *There is a one-to-one correspondence between relative zero-cycles as of Definition* (2.3)*, and relative cycles of dimension zero, as of Definition* (4.2)*. In this correspondence the support remains the same and $\alpha_{U/B/T} = \alpha_{U/B}$.*

Proof. As $\alpha_{U/B/T} = \alpha_{U/B/B}$ by Proposition (4.11), this correspondence is well-defined. Under the hypothesis that $\alpha_{U/B/T} = \alpha_{U/B/B}$, it is then enough to check the compatibility condition in the second definition for morphisms between projections of the form $(U_1, B_1, B_1) \to (U_2, B_2, B_2)$. This compatibility condition coincides with the compatibility condition between neighborhoods $(U_1, B_1) \to (U_2, B_2)$ in the first definition. $\square$

Definition (4.13). We let $\mathit{Cycl}(X/S)$ be the set of relative cycles on X/S. We let $\mathit{Cycl}_{\mathrm{equi}}(X/S)$ (resp. $\mathit{Cycl}_r(X/S)$, resp. $\mathit{Cycl}^{\mathrm{prop}}(X/S)$, resp. $\mathit{Cycl}^{\mathrm{cl}}(X/S)$) be the subset consisting of relative cycles which are equidimensional (resp. are equidimensional of dimension r, resp. have proper support, resp. have closed support). We let $\mathrm{Chow}_r(X/S)$ and $\mathrm{Chow}(X/S)$ be the functors from S-schemes to sets given by

$$\begin{aligned} \mathrm{Chow}_r(X/S)(T) &= \mathit{Cycl}_r^{\mathrm{prop}}(X \times_S T/T) \\ \mathrm{Chow}(X/S)(T) &= \mathit{Cycl}_{\mathrm{equi}}^{\mathrm{prop}}(X \times_S T/T) \end{aligned}$$

with the natural pull-back.

As before it follows that $\mathit{Cycl}(X/S)$, $\mathrm{Chow}(X/S)$, $\mathrm{Chow}_r(X/S)$, etc., are fppf-sheaves as $\Gamma^\star(U/B)$ is representable.

Definition (4.14). Let X/S be locally of finite type. We say that a relative cycle α on X/S is a *relative Weil divisor* if for every $s \in S$ and $z \in \mathrm{Supp}(\alpha)_s$ we have that $\mathrm{codim}_z(\mathrm{Supp}(\alpha)_s, X_s) = 1$.

(4.15) *Addition of cycles* — Let α and β be relative cycles on X/S with supports Z_α and Z_β. If Z_α and Z_β are closed in $Z_\alpha \cup Z_\beta$, e.g., if Z_α and Z_β are closed in X, then there is a relative cycle $\alpha + \beta$ on X/S with support $Z_\alpha \cup Z_\beta$ defined by $(\alpha+\beta)_{U/B/T} = \alpha_{U/B/T} + \beta_{U/B/T}$ for any projection adapted to $Z_\alpha \cup Z_\beta$, cf. (2.7). This makes $\mathit{Cycl}^{\mathrm{cl}}(X/S)$ a commutative monoid.

5. Smooth projections

In this section, we show that in the definition of a relative cycle, given in the previous section, it is enough to consider smooth projections. That is, relative zero-cycles on every smooth projection satisfying the compatibility condition, determine a unique relative cycle. We then discuss variants of the definition of a relative cycle that are more well-behaved.

Lemma (5.1) ([EGA$_{\mathrm{IV}}$, Prop. 18.1.1])**.** *Let $S_0 \hookrightarrow S$ be a closed immersion. Let $X_0 \to S_0$ be smooth (resp. étale) and $x_0 \in X_0$. Then there is an open neighborhood $U_0 \subseteq X_0$ of x_0 and a smooth (resp. étale) scheme $U \to S$ such that $U_0 = U \times_S S_0$.*

Lemma (5.2). *Let $S_0 \hookrightarrow S$ be a closed immersion. Let X/S be a scheme and let Y/S be smooth. Let $X_0 = X \times_S S_0$, let $x_0 \in X_0$ and let $f_0 : X_0 \to Y$ be a morphism. Then there exists an open neighborhood $U_0 \subseteq X_0$, an étale morphism $U \to X$ such that $U_0 = U \times_X X_0$, and a map $f : U \to Y$ which restricts to $(f_0)|_{U_0}$.*

Proof. Replacing X and Y with open neighborhoods, we can assume that Y/S factors through an étale map $Y \to \mathbb{A}^n_S$. As f_0 lifts to $X \to \mathbb{A}^n_S$, we can replace S with $\mathbb{A}^n_S$ and assume that Y/S is étale. Let $V = X \times_S Y$ and

$V_0 = V \times_X X_0 = X_0 \times_S Y$. Then $V_0 \to X_0$ has an open section s. Any open subset $U \subseteq V$ restricting to $s(X_0)$ gives a map as in the lemma. □

Proposition (5.3). *In the definition of relative cycles, Definition* (4.2)*, it is enough to consider projections* (U, B, T) *such that* $T = \mathbb{A}^n_S$ *for some* n*. That is, given* Z *as in the definition and relative cycles* $\alpha_{U/B/T}$ *on projections* (U, B, T) *with* $T = \mathbb{A}^n_S$ *for some* n *satisfying the compatibility condition, these data extends uniquely to a relative cycle.*

Proof. It is clear that in the definition of relative cycles, we can assume that U, B and T are affine. We can also assume that S and X are affine. Let (U, B, T) be a projection. Then T is the inverse limit of finitely presented affine S-schemes T_λ. As $U/X \times_S T$, and B/T are of finite presentation, it follows that the projection descends to a projection $(U_\lambda, B_\lambda, T_\lambda)$ for sufficiently large λ. Similarly, every morphism of neighborhoods $(U_1, B_1, T) \to (U_2, B_2, T)$ descends to a morphism of finitely presented neighborhoods. In the definition of relative cycles, we can thus assume that all projections are finitely presented.

Let (U, B, T) be a projection with T a finitely presented affine S-scheme. There is then a closed immersion $T \hookrightarrow T_1 = \mathbb{A}^n_S$. Lemmas (5.1) and (5.2) shows that, locally on U and B, there exists an étale morphism $U_1 \to X_1 \times_S T_1$, a smooth morphism $B_1 \to T_1$ and a morphism $U_1 \to B_1$, lifting $U \to X \times_S T$, $B \to T$ and $U \to B$ respectively.

To show that $\alpha_{U/B/T}$ is uniquely defined by smooth projections, we can assume that $B = \mathbb{A}^n_T$. Let (U_1, B_1, T_1) and (U_2, B_2, T_2) be two smooth liftings, i.e., $T_i = \mathbb{A}^{n_i}_S$, $B_i = \mathbb{A}^r_{T_i}$ and $(U_i, B_i, T_i) \times_{T_i} T = (U, B, T)$. Then $T \to T_2$ (resp. $B \to B_2$) factors non-canonically through $T \to T_1$ (resp. $B \to B_1$). Replacing U_1 with an étale cover, we can also arrange so that $U \to U_2$ factors through $U \to U_1$. Thus, if the smooth projections are compatible, then $\alpha_{U/B/T}$ is uniquely defined by them.

Finally, let us show that the compatibility condition for smooth projections imply the compatibility condition for arbitrary projections. As the $\alpha_{U/B/T}$'s are compatible with base change by assumption, it is enough to check the compatibility for $(U, B_1, T) \to (U, B_2, T)$ and $(U_1, B, T) \to (U_2, B, T)$. By Lemmas (5.1) and (5.2), these morphisms lift to morphisms of projections over $\mathbb{A}^n_S$. □

Corollary (5.4). *Let* $Z \hookrightarrow X$ *be a locally closed subset, universally open over* S*, and assume that we are given relative zero-cycles* $\alpha_{U/B/S}$ *for every projection* (U, B, S) *adapted to* Z*. Then there is at most one relative cycle inducing these relative zero-cycles.*

Proof. By Proposition (5.3) a relative cycle α is given by its smooth projections. By Corollary (B.3), a relative cycle α is determined by its étale projections. Finally if (U, B, T) is an étale projection, then $\alpha_{U/B/T} = \alpha_{U/B/S}$ by Proposition (4.11). □

A relative cycle on X/S is expected to behave as if it is induced by an object living on X. Thus, the following condition is reasonable.

(*) For any smooth projection (U, B, T) the relative zero-cycle $\alpha_{U/B/T}$ does not depend on T.

We will show that this is satisfied in many situations, cf. Proposition (9.17). I do not know if this condition always holds for a relative cycle but this seems unlikely. If not, then this condition should probably be imposed on relative cycles to get a well-behaved functor, cf. Section 16.

Moreover, it is also reasonable to require that for any pair of smooth morphisms $p : U \to X$ and $B \to S$ and a morphism $U \to B$ such that $U \to B$ is quasi-finite over $p^{-1}(Z)$, there is a relative zero-cycle $\alpha_{U/B}$ on U/B. Indeed, if smooth pull-back of relative cycles exists, then such relative zero-cycles $\alpha_{U/B}$ exist. This is the case if S is reduced, cf. Section 14. I do not know if this follows in general from condition $(*)$.

Given a relative cycle α on X/S, it is also fair to require that there should be an infinitesimal neighborhood Z of $\mathrm{Supp}(\alpha)$ such that α is the push-forward of a relative cycle on Z/S. If α is a relative zero-cycle, then there is the canonical choice $Z = \mathrm{Image}(\alpha)$. The following proposition gives sufficient and necessary conditions for the existence of an infinitesimal neighborhood Z as above.

Proposition (5.5). *Let α be a relative cycle on X/S with support $Z_0 \subseteq X$. Let $Z_0 \hookrightarrow Z$ be an infinitesimal neighborhood. Then α is the push-forward of a relative cycle on Z if and only if*

(i) *For any smooth projection (U, B, T, p) adapted to Z_0, the image of $\alpha_{U/B/T}$ is contained in $p^{-1}(Z)$.*
(ii) *For any smooth projection (U, B, T, p) adapted to Z_0, the relative cycle $\alpha_{U/B/T}$ only depends upon $U|_{p^{-1}(Z)} \to B$ and $p|_Z$.*

Proof. The two conditions are clearly necessary. To show that they are sufficient it is enough to show that given a smooth projection (U, B, T, p) of Z/S adapted to Z_0, there is a smooth projection (U', B, T, p') of X/S adapted to Z_0 which restricts to the first projection over Z, and similarly for morphisms of projections. This follows from Lemmas (5.1) and (5.2). $\square$

6. Uniqueness and extension of relative cycles

Proposition (6.1). *Let S be an irreducible normal scheme with generic point ξ and X/S locally of finite type. Let $Z \hookrightarrow X$ be a subscheme such that Z/S is equidimensional of dimension zero, i.e., locally quasi-finite and such that $Z_{\mathrm{dom}/S} = Z$. Then any relative cycle on $X_\xi/\mathrm{Spec}(k(\xi))$ with support Z_ξ extends uniquely to a relative cycle on X/S.*

Proof. If $g : T \to S$ is étale then T is normal. As it is enough to consider étale neighborhoods (U, T, p, g) in the definition of a relative non-proper cycle, we can thus assume that Z/S is finite. Let α_ξ be a relative cycle on $X_\xi/\mathrm{Spec}(k(\xi))$ and let W_ξ be its image, which is an infinitesimal neighborhood of Z_ξ. Let $W \hookrightarrow X$ be the closure of W_ξ. Then as $\Gamma^d(W/S) \to S$ is finite [Ryd08a, Prop. 4.3.1] and S is normal, it follows that the morphism $\alpha_\xi : \mathrm{Spec}(k(\xi)) \to \Gamma^d(W/S)$ extends to a section of $\Gamma^d(W/S) \to S$. $\square$

Corollary (6.2). *Let S be an irreducible normal scheme with generic point ξ and X/S locally of finite type. Let $Z \hookrightarrow X$ be a subscheme satisfying* (T). *Then any relative cycle on $X_\xi/\mathrm{Spec}(\xi)$ with support Z_ξ extends uniquely to a relative cycle on X/S.*

Proof. Follows from Proposition (6.1) as it is enough to consider smooth projections. □

Chevalley's criterion for universally open morphisms [$\mathrm{EGA}_{\mathrm{IV}}$, Thm. 14.4.1] easily follows. Note that the proof given in *loc. cit.* only is valid if X/S is locally of finite presentation [$\mathrm{EGA}_{\mathrm{IV}}$, $\mathrm{Err}_{\mathrm{IV}}$ 37].

Corollary (6.3) (Chevalley's theorem)**.** *Let S be a geometrically unibranch scheme (e.g. a normal scheme) with a finite number of components and let $X \to S$ be locally of finite type satisfying* (T)*, i.e., such that every point $x \in X$ is contained in an irreducible component which is equidimensional over S at x. Then $X \to S$ is universally open.*

Proof. Let $\widetilde{S} \to S$ be the normalization. As this is a universal homeomorphism, we can assume that S is normal. We will now construct a canonical relative cycle α on X/S with support X. The underlying cycle, cf. Section 8, of α is going to be $[X]$. Let (U, B, T) be any smooth projection. Then B is normal and U/B is generically flat. We let $\alpha_{U/B/T}$ be the unique extension of $\mathcal{N}_{U_\xi/B_\xi}$ given by Proposition (6.1). The corollary then follows from Proposition (4.7). □

Note that the condition that S has a finite number of components is essential. In fact, there are non-noetherian normal schemes such that the irreducible components are not open, e.g., the absolutely flat scheme associated to the affine line. The inclusion of such a component is a counter-example.

We have the following simple analog of the flatification by Raynaud and Gruson [RG71]:

Proposition (6.4). *Let S be a scheme, X/S locally of finite type and let $U \subseteq S$ be an open retro-compact subset. Let $Z \hookrightarrow X$ be a subscheme such that Z/S is universally open. Let α_U be a relative cycle on $X|_U/U$ with support $Z|_U$. Let $S' \to S$ be the normalization of S in U, i.e., the spectrum of the integral closure of $\mathcal{O}_S$ in the direct image of $\mathcal{O}_U$. Then α_U extends to a relative cycle on X'/S'.*

Proof. As the integral closure commutes with smooth morphisms, we can assume that Z/S is zero-dimensional. Then reason as in the proof of Proposition (6.1). □

Proposition (6.5). *Let S be a locally noetherian scheme, X/S locally of finite type and let $U \subseteq S$ be an open subscheme. Let $Z \hookrightarrow X$ be a subscheme such that Z/S satisfies* (T)*. Let α_U be a relative cycle on $X|_U/U$ with support $Z|_U$.*

(i) *If U contains all points of depth zero, then there is at most one relative cycle on X/S extending α_U.*
(ii) *If U contains all points of depth at most one, then there is a unique relative cycle on X/S extending α_U.*

Proof. If $B \to S$ is flat and $U \subseteq S$ contains all points of depth zero (resp. at most one) then so does $B \times_S U \subseteq B$. As it is enough to consider smooth projections, we can thus assume that Z/S is finite. Then α_U is a relative proper zero-cycle and we let $W \hookrightarrow X$ be its image. If U contains all points of depth zero, then the morphism $U \to \Gamma^d(W/S)$ has at most one extension to S. If U contains all points of depth one, then as $\Gamma^d(W/S) \to S$ is finite and in particular affine, it follows that the section $U \to \Gamma^d(W/S)$ extends to S. Indeed, if $j : U \hookrightarrow S$ is the inclusion, then $j_*\mathcal{O}_U = \mathcal{O}_S$. □

We can make the extension property slightly more precise.

Corollary (6.6). *Let S be a locally noetherian scheme, let $f : X \to S$ be locally of finite type and let $U \subseteq X$ be an open subscheme. Let $Z \hookrightarrow X$ be a subscheme such that Z/S satisfies* (T)*. Let α_U be a relative cycle on U/S with support $Z|_U$.*

- (i) *If U contains all points $z \in Z$ such that* $\operatorname{depth} f(z) + \operatorname{codim}_z(Z_{f(z)}) = 0$*, then there is at most one relative cycle on X/S with support Z extending α_U.*
- (ii) *If U contains all points $z \in Z$ such that* $\operatorname{depth} f(z) + \operatorname{codim}_z(Z_{f(z)}) \leq 1$*, then there is a unique relative cycle on X/S with support Z extending α_U.*

Proof. This follows from Proposition (6.5) and the observation that if $h : B \to S$ is smooth, then the depth of a point $b \in B$ is the sum of the depth of $h(b)$ and the codimension of b in its fiber $h^{-1}(h(b))$. □

7. Flat families

In this section, we will define a relative cycle $\mathcal{N}_{\mathcal{F}/S}$ on X/S for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ which is flat over S. If (U, B, T, p) is a projection, then $p^*\mathcal{F}$ is not flat over B, but only of *finite Tor-dimension*, cf. Lemma (7.12). If for every point $s \in S$ of depth zero, $\mathcal{F}_s$ has no embedded components in codimension one, then $p^*\mathcal{F}/B$ is flat at every point of depth one and the existence of $(\mathcal{N}_{\mathcal{F}/S})_{U/B/T}$ follows from Proposition (6.5). In general, however, we need to associate a relative zero-cycle to a coherent sheaf of finite Tor-dimension. Similar constructions can be found in [GIT, Ch. 5, §3], [Fog69, §2] and [KM76]. To avoid complicated notions such as pseudo-coherence, we only use the notion of finite Tor-dimension for coherent modules over noetherian schemes.

Definition (7.1) ([SGA$_6$, Exp. I, Def. 5.2])**.** Let $(X, \mathcal{A})$ be a locally ringed space. An $\mathcal{A}$-module $\mathcal{F}$ on X has *finite Tor-dimension* if $\mathcal{F}_x$ is an $\mathcal{A}_x$-module of finite Tor-dimension for every $x \in X$, i.e., if $\mathcal{F}_x$ admits a finite resolution of flat $\mathcal{A}_x$-modules. The Tor-dimension of $\mathcal{F}$ at x, denoted $\operatorname{Tor-dim}_x(\mathcal{F})$, is the length n of a minimal flat resolution of $\mathcal{A}_x$-modules

$$0 \to \mathcal{P}_n \to \mathcal{P}_{n-1} \to \cdots \to \mathcal{P}_0 \to \mathcal{F}_x \to 0.$$

Note that $\mathcal{F}$ is flat at x if and only if the Tor-dimension of $\mathcal{F}$ at x is zero.

Definition (7.2) ([SGA$_6$, Exp. III, Def. 3.1])**.** Let $f : X \to S$ be a morphism of algebraic spaces and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The module $\mathcal{F}$ has *finite Tor-dimension* over S if $\mathcal{F}$ has finite Tor-dimension as a $f^{-1}\mathcal{O}_S$-module.

Remark **(7.3).** If $f : X \to S$ is affine and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then $\mathcal{F}$ is of finite Tor-dimension over S if and only if $f_*\mathcal{F}$ is of finite Tor-dimension.

(7.4) *Auslander-Buchsbaum formula* — Let X be a locally noetherian scheme and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module of finite Tor-dimension. Then $\operatorname{Tor-dim}_x(\mathcal{F}) + \operatorname{depth}_x(\mathcal{F}) = \operatorname{depth} x$ [AB57, Thm. 3.7]. In particular, $\mathcal{F}$ is flat, and hence free, over points of depth zero. If $\mathcal{F}$ is $(\mathrm{S_k})$, then $\mathcal{F}$ is flat over points of depth at most k.

(7.5) *Norms and traces* — Let A be a ring, B an A-algebra and M a B-module which is locally free of rank d as an A-module. Then the norm map of M, which defines $\mathcal{N}_M$, is given by

$$B \longrightarrow \operatorname{End}_A(M) \xrightarrow{\det} \operatorname{End}_A(\wedge^d M) \cong A$$

where the first homomorphism is the multiplication, and the second *map* takes an endomorphism $\varphi \in \operatorname{End}_A(M)$ onto the endomorphism $\wedge^d\varphi$ given by

$$x_1 \wedge \cdots \wedge x_d \mapsto \varphi(x_1) \wedge \cdots \wedge \varphi(x_d).$$

We also have a *trace* homomorphism given by a similar composition where the second map is the *homomorphism* which takes φ onto the endomorphism

$$x_1 \wedge \cdots \wedge x_n \mapsto \varphi(x_1) \wedge x_2 \wedge \cdots \wedge x_n + x_1 \wedge \varphi(x_2) \wedge \cdots \wedge x_n + \cdots + x_1 \wedge x_2 \wedge \cdots \wedge \varphi(x_n).$$

Now assume that M is not locally free but of finite Tor-dimension. Let $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ be a locally free resolution. If $\varphi \in \operatorname{End}_A(M)$ then, as the P_i's are projective, there is a (non-unique) lifting of the endomorphism φ to an endomorphism $\varphi_\bullet$ of the complex $P_\bullet$. The trace of φ on M, can then be defined as the alternating sum $\sum_i (-1)^i \operatorname{tr}_{P_i}(\varphi_i)$. If M is locally free, the resolution splits locally and it is clear that this definition of the trace of M coincides with the previous definition. It thus follows that the trace of an arbitrary M of finite Tor-dimension is independent of the resolution and the choice of lifting $\varphi_\bullet$. In fact, M is free over every point in $\operatorname{Spec}(A)$ of depth zero.

Naïvely, we would define the norm of a module of finite Tor-dimension similarly, i.e., $\prod_i \mathrm{N}_{P_i}(\varphi_i)^{(-1)^i}$, but this does not make sense unless $\mathrm{N}_{P_i}(\varphi_i)$ is invertible for every odd i. The following easy lemma, similar to Gauss's Lemma, solves this.

Lemma (7.6). *Let $A \hookrightarrow A'$ be a ring extension. Let $p, q \in A[t]$ and $r \in A'[t]$ be monic polynomials. If $rp = q$ in $A'[t]$ then $r \in A[t]$.*

Lemma (7.7). *Let A be a noetherian ring and let*

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

be an exact sequence of A-modules such that the P_i's are free of finite ranks. Let $\varphi \in \operatorname{End}_A(M)$. Then there is a unique element, $\det(\varphi) \in A$, coinciding with the usual determinant of φ at any point $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $M_\mathfrak{p}$ is free.

Proof. First note that the uniqueness of $\det(\varphi)$ is clear as $M_{\mathfrak{p}}$ is free over any point $\mathfrak{p}$ of depth zero. Let $\varphi_i \in \operatorname{End}_A(P_i)$, be liftings of the endomorphism φ. Let $\varphi' \in \operatorname{End}_{A[t]} M[t]$ and $\varphi'_i \in \operatorname{End}_{A[t]} P_i[t]$ be defined by

$$\varphi' = \mathrm{id}_M \otimes t + \varphi \otimes \mathrm{id}_{A[t]}$$
$$\varphi'_i = \mathrm{id}_{P_i} \otimes t + \varphi_i \otimes \mathrm{id}_{A[t]}$$

where t also denotes multiplication by t. Then $\det(\varphi'_i)$ — the characteristic polynomial of φ_i — is a monic polynomial for all i. It is enough to show the existence of $\det(\varphi')$.

Let $\operatorname{Tot}(A)$ be the total ring of fractions of A, i.e., the localization in the set of all regular elements. Recall that $\operatorname{Tot}(A)$ is a semi-local ring such that every maximal ideal has depth zero. It follows that $M \otimes_A \operatorname{Tot}(A)$ is locally free of rank d, and hence free [Bou61, Ch. II, §2.3, Prop. 5]. Thus, there exists a regular element $f \in A$ such that M_f is free.

Let $p = \prod_{2\nmid i} \det(\varphi'_i) \in A[t]$, $q = \prod_{2|i} \det(\varphi'_i) \in A[t]$ and $r = \det(\varphi'_f) \in A_f[t]$. Then $rp = q$ in $A_f[t]$ and hence $p \in A[t]$ by the lemma. The element $p(0) \in A$ is the determinant of φ. □

Proposition (7.8). *Let S be a locally noetherian space and let $f : X \to S$ be a morphism of algebraic spaces, locally of finite type. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module such that* $\operatorname{Supp}(\mathcal{F})$ *is finite over S and $\mathcal{F}$ has finite Tor-dimension over S. Then there is a unique proper relative zero-cycle $\mathcal{N}_{\mathcal{F}/S} : S \to \Gamma^\star(X/S)$ on X/S such that for any point $s \in S$ of depth zero, the induced cycle $\left(\mathcal{N}_{\mathcal{F}/S}\right)_s$ is given by the norm $\mathcal{N}_{(f_*\mathcal{F})_s}$ of the free $\mathcal{O}_{S,s}$-module $(f_*\mathcal{F})_s$. In particular, the degree of $\mathcal{N}_{\mathcal{F}/S}$ at $s \in S$ is the rank of $\mathcal{F}$ over any generization of s and the support of $\mathcal{N}_{\mathcal{F}/S}$ is the closure of the support of* $\operatorname{Supp}(\mathcal{F})$ *over the generic points. This construction commutes with cohomologically flat base change, i.e., base change $S' \to S$ such that* $\operatorname{Tor}_i^S(\mathcal{O}_{S'}, \mathcal{F}) = 0$ *for all $i > 0$.*

Proof. Let $\mathcal{I} = \operatorname{Ann}_{\mathcal{O}_X}(\mathcal{F})$ be the annihilator of $\mathcal{F}$ and let $j : Z \hookrightarrow X$ be the closed subscheme defined by $\mathcal{I}$. Then $Z \to S$ is finite and $\mathcal{F} = j_*j^*\mathcal{F}$. Replacing X with Z, we can thus assume that f is finite. The norm

$$f_*\mathcal{O}_X \longrightarrow \operatorname{End}_{\mathcal{O}_S}(f_*\mathcal{F}) \xrightarrow{\det} \mathcal{O}_S$$

defines a multiplicative law and hence a proper relative zero-cycle $\mathcal{N}_{\mathcal{F}/S}$ as in the proposition. □

Corollary (7.9). *Let S be locally noetherian and let $f : X \to S$ be a morphism locally of finite type. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module of finite type such that* $\operatorname{Supp}(\mathcal{F})$ *is quasi-finite over S and such that $\mathcal{F}$ has finite Tor-dimension over S. Then there is a unique relative zero-cycle $\mathcal{N}_{\mathcal{F}}$ on X/S with support* $\operatorname{Supp}(\mathcal{F})_{\mathrm{dom}/S}$ *such that for any point $s \in S$ of depth zero, the induced cycle $(\mathcal{N}_{\mathcal{F}})_s$ is given by the norm $\mathcal{N}_{(f_*\mathcal{F})_s}$ of the free $\mathcal{O}_{S,s}$-module $(f_*\mathcal{F})_s$. This construction commutes with cohomologically flat base change.*

Proof. Proposition (7.8) gives a unique proper relative zero-cycle $\alpha_{U/T}$ on any étale neighborhood (U, T, p, g) which thus determines the relative zero-cycle $\mathcal{N}_{\mathcal{F}}$ by Corollary (2.20). □

Remark **(7.10).** Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be coherent $\mathcal{O}_X$-modules of finite Tor-dimension over S and with quasi-finite support over S. The following properties of the norm of a sheaf of finite Tor-dimension are easily verified.

(i) If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-sheaf, then $\mathcal{N}_{\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}} = \mathcal{N}_{\mathcal{F}}$.
(ii) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence, then $\mathcal{N}_{\mathcal{G}} = \mathcal{N}_{\mathcal{F}} + \mathcal{N}_{\mathcal{H}}$. In particular, we have that $\mathcal{N}_{\mathcal{F}\oplus\mathcal{G}} = \mathcal{N}_{\mathcal{F}} + \mathcal{N}_{\mathcal{G}}$.

Remark **(7.11).** *Norms of perfect complexes* — There is an analog of Proposition (7.8) for certain perfect complexes. Note that not every perfect complex determines a relative cycle. Indeed, a necessary condition is that it is possible to define a relative cycle on depth zero points, i.e., that the alternating determinant is defined on depth zero points. This is also a sufficient condition by the proof of Lemma (7.7).

If $\mathcal{F}_\bullet$ is a perfect complex on S such that the norm of $\mathcal{F}_\bullet$ is defined and $\bigoplus_i \mathrm{H}_i(S, \mathcal{F}_\bullet)$ is of finite Tor-dimension and zero in odd degree, then we have that the norms of $\mathcal{F}_\bullet$ and $\bigoplus_i \mathrm{H}_i(S, \mathcal{F}_\bullet)$ coincide. In particular, if $\mathcal{F}_\bullet$ is a perfect complex on S such that at depth zero points, $\mathrm{H}_i(S, \mathcal{F}_\bullet)$ is zero for odd i and locally free for even i, then the norm of $\mathcal{F}_\bullet$ is defined.

Lemma (7.12) ([GIT, Lem. 5.8])**.** *Let $h : B \to S$ be smooth and let $\varphi : X \to B$ be locally of finite type. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module which has finite Tor-dimension over S, then $\mathcal{F}$ has finite Tor-dimension over B.*

Proof. Consider the product $X \times_S B$. The first projection π_1 has a section $s = (\mathrm{id}_X, \varphi) : X \to X \times_S B$ which is a regular immersion. Thus, $\mathcal{O}_X$ has finite Tor-dimension over $X \times_S B$. The pull-back $\pi_1^*\mathcal{F}$ has finite Tor-dimension over B and thus $\mathcal{F} = s^*\pi_1^*\mathcal{F}$ has finite Tor-dimension over B. $\square$

Proposition (7.13). *Let S be locally noetherian and let $f : X \to S$ be a smooth curve. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module of finite Tor-dimension over S and such that $Z = \mathrm{Supp}(\mathcal{F})$ is quasi-finite over S. Then $\mathcal{N}_{\mathcal{F}/S} = \mathcal{N}_{\mathrm{Div}(\mathcal{F})/S}$ where $\mathrm{Div}(\mathcal{F})$ is the relative Cartier-divisor on X defined by Mumford* [GIT, KM76].

Proof. Note that $\mathcal{F}$ has finite Tor-dimension as an $\mathcal{O}_X$-module by Lemma (7.12) and thus $\mathrm{Div}(\mathcal{F})$ is defined. The support of both $\mathcal{N}_{\mathcal{F}/S}$ and $\mathcal{N}_{\mathrm{Div}(\mathcal{F})/S}$ is $Z_{\mathrm{dom}/S}$. By Proposition (6.5) it is enough to show the equality on depth zero points. Taking an étale neighborhood, we can thus assume that Z/S is finite and that $f_*\mathcal{F}$ is a *free* $\mathcal{O}_S$-module. The equality now follows from [Del73, Prop. 6.3.11.1]. $\square$

Theorem (7.14). *Let $f : X \to S$ be a morphism locally of finite presentation and let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module which is flat over S. Then there is a canonical relative cycle, denoted $\mathcal{N}_{\mathcal{F}}$ on X/S with support* $\mathrm{Supp}(\mathcal{F})$*. This construction commutes with arbitrary base change. If $Z \hookrightarrow X$ is a subscheme such that Z is flat and of finite presentation over S, we let $\mathcal{N}_Z = \mathcal{N}_{\mathcal{O}_Z}$.*

Proof. The question is local so we can assume that X and S are affine. By a limit argument, we can then assume that S is noetherian. First note that the support Z of $\mathcal{F}$ is universally open [EGA$_{\mathrm{IV}}$, 2.4.6]. Let (U, B, T) be a projection adapted to Z. Then $\mathcal{F}$ is of finite Tor-dimension over B and we let $(\mathcal{N}_\mathcal{F})_{U/B/T} = \mathcal{N}_{\mathcal{F}/B}$. This defines a relative cycle. Note that any base change $B \times_S S' \to B$ is cohomologically flat with respect to $\mathcal{F}$ over B as $\mathcal{F}$ is flat over S. □

Note that $\mathcal{N}_\mathcal{F}$ is defined for sheaves $\mathcal{F}$ with non-proper and non-equidimensional support. We do not even require that X/S is separated. For representability, we need families of cycles to be equidimensional. However, even if $\mathcal{F}$ is a sheaf whose support is not equidimensional, then we have the equidimensional relative cycle $(\mathcal{N}_\mathcal{F})_r$. If $\mathcal{F}$ has proper support with fibers of dimension at most r, then $(\mathcal{N}_\mathcal{F})_r$ is a proper relative cycle of dimension r. In particular, we obtain the following morphism:

Corollary (7.15). *There is a canonical morphism from the functor* $\mathrm{Quot}_r(\mathcal{G}/X/S)$ *to the functor* $\mathrm{Chow}_r(X/S)$ *given by* $\mathcal{F} \mapsto (\mathcal{N}_\mathcal{F})_r$. *Similarly, there is a canonical morphism from the functor* $\mathrm{Hilb}_r(X/S)$ *to the functor* $\mathrm{Chow}_r(X/S)$ *given by* $Z \mapsto (\mathcal{N}_Z)_r$. *Here* $\mathrm{Hilb}_r(X/S)$ *is the Hilbert functor parameterizing subschemes Z which are proper and of dimension r but not necessary equidimensional, and* $\mathrm{Chow}_r(X/S)$ *is the Chow functor parameterizing equidimensional proper relative cycles of dimension r.*

Later on, we will see that there also are morphisms from the Hilbert stack and from the Kontsevich space of stable maps to the Chow functor. In particular, we obtain a morphism from the stack of Branch varieties [AK06] and from the space of Cohen-Macaulay curves [Høn04] to the Chow functor. This is discussed in Section 13.

8. The underlying cycle

In this section we will assign to any relative cycle α on X/S, an ordinary cycle, denoted $\mathrm{cycl}(\alpha)$. The support of $\mathrm{cycl}(\alpha)$ coincides with the support of α. As the support is universally open, the only thing that we need to define is the multiplicities of the components over a generic point of S.

Proposition-Definition (8.1). *Let $S = \mathrm{Spec}(k)$ be the spectrum of a field and α be a relative cycle on X/S. Let $x \in X$ be a point which is generic in* $\mathrm{Supp}(\alpha)$. *Then there is a unique number* $\mathrm{mult}_x(\alpha)$, *the* multiplicity *of α at x, such that for any projection (U, B, T, p, g) and $u \in U$ above $x \in X$ and $t \in T$ such that $k(t)/k(s)$ is* separable, *we have that* $\mathrm{mult}_u(\alpha_{U/B/T}) = \mathrm{mult}_x(\alpha)$, *cf. Definition* (1.10). *The* geometric multiplicity *at x is the product of the multiplicity at x and the radical multiplicity of* $k(x)/k(s)$ [EGA$_{\mathrm{IV}}$, Def. 4.7.4]. *The geometric multiplicity is constant under arbitrary base change.*

Proof. We first observe that if (U, B, T) is a projection, $T' \to T$ is an arbitrary morphism and $u' \in U' = U \times_T T'$ a point above u, then the geometric multiplicities of $\alpha_{U/B/T}$ at u and $\alpha_{U'/B'/T'}$ at u' coincide. This is Proposition (1.13, (ii)). Thus, $\mathrm{mult}_{u'}(\alpha_{U'/B'/T'}) = \mathrm{mult}_u(\alpha_{U/B/T})r$ where r is the length of $\mathrm{Spec}(k(u) \otimes_{k(t)} k(t'))$ at u' [EGA$_{\mathrm{IV}}$, Prop. 4.7.3]. It follows that the multiplicites at u' and u, multiplied with the radical multiplicity of $k(u')/k(t')$ and $k(u)/k(t)$ respectively, coincide.

It is thus enough to show that if $\overline{k}$ is an algebraically closed field, $(U_1, B_1, \mathrm{Spec}(\overline{k}))$ and $(U_2, B_2, \mathrm{Spec}(\overline{k}))$ are two projections and $u_1 \in U_2$ and $u_2 \in U_2$ are two points above x, then the multiplicites of $\alpha_{U_1/B_1/\overline{k}}$ at u_1 and $\alpha_{U_2/B_2/\overline{k}}$ at u_2 coincide. As the multiplicity is constant under pull-back by étale morphisms $U' \to U_i$ by Proposition (1.13, (iv)), we can replace the U_i's with $U_1 \times_X U_2$ and the u_i's with (u_1, u_2) and hence assume that $X = U_1 = U_2$ and $x = u$. Taking étale projections $B_1 \to \mathbb{A}^r$ and $B_2 \to \mathbb{A}^r$, which is possible locally around the images of x and using Proposition (1.13, (vi)) we can assume that $B_1 = B_2 = \mathbb{A}^r$. Taking an open neighborhood of x, we can assume that $Z = \operatorname{Supp}(\alpha)$ is smooth and irreducible.

Let φ_1 and φ_2 be the two projections $X \to \mathbb{A}^r$. It is enough to show that the multiplicity of α_{φ_1} at x coincides with the multiplicity of α_{φ_2} for a particular choice of φ_2. Taking a generic projection, we can thus assume that $\varphi_2|_Z$ is étale. The morphisms φ_1 and φ_2 can be put into a single projection

$$\varphi : X \times_{\overline{k}} T \to \mathbb{A}^r \times_{\overline{k}} T$$

over $T = \mathbb{A}^1_{\overline{k}}$ such that $\varphi_1 = \varphi|_{t=0}$ and $\varphi_2 = \varphi|_{t=1}$. Let $U \subseteq X \times T$ be the open subset where $\varphi|_{Z\times T}$ is quasi-finite. This subset contains $X \times \{0\}$ and $X \times \{1\}$. As $Z \times T \to T$ is Cohen-Macaulay it follows that $\varphi|_{Z\times T}$ is flat over U. Moreover, as $\varphi_2|_Z$ is étale it follows that $\varphi|_{Z\times T}$ is generically étale. It then readily follows from [Ryd08b, Prop. 8.6] that the (non-proper) relative zero-cycle α_φ is of the form $m \cdot \mathcal{N}_{Z\times T/\mathbb{A}^r\times T}$ for some positive integer m. We thus have that $\alpha_{\varphi_i} = m\mathcal{N}_{Z/_{\varphi_i}\mathbb{A}^r}$ for $i = 1, 2$. It follows that $\operatorname{mult}_x \alpha_{\varphi_1} = \operatorname{mult}_x \alpha_{\varphi_2} = m$. □

Definition (8.2). Let S be arbitrary and let α be a relative cycle on X/S with support Z. The *underlying cycle* of α is the effective cycle $\operatorname{cycl}(\alpha)$ with $\mathbb{Q}$-coefficients defined by

$$\operatorname{cycl}(\alpha) = \sum_{\substack{s\in S_{\max}\\ x\in (Z_s)_{\max}}} \operatorname{mult}_x(\alpha_s)\left[\overline{\{x\}}\right].$$

Here $\overline{\{x\}}$ denotes the closure of x in $\operatorname{Supp}(\alpha)$ as a reduced subscheme.

Remark **(8.3).** It follows from Proposition-Definition (8.1) that if (U, B, T, p) is a smooth projection, then $p^* \operatorname{cycl}(\alpha) = \operatorname{cycl}(\alpha_{U/B/T})$.

The following definition generalizes [Ryd08b, Def. 8.1].

Definition (8.4). Let K/k be a finitely generated field extension. The *inseparable degree*, or radical multiplicity [$\mathrm{EGA_{IV}}$, Def. 4.7.4], is the maximum length of $K \otimes_k k'$ where k'/k is an inseparable extension. The *exponent* of K/k is the smallest integer e such that $K^e k/k$ is separable. The *inseparable discrepancy* is the quotient of the inseparable degree and the exponent.

If K/k is a finitely generated field extension and $k' = k(x_1, x_2, \dots, x_r) \subseteq K$ is a transcendence basis, then the exponent of K/k' is a multiple of the exponent of K/k. Moreover, there is a transcendence basis such that the exponent of K/k' equals the exponent of K/k, e.g., take k' as a separating transcendence basis of $K^e k/k$.

Definition (8.5). Let S be a scheme and let X/S be locally of finite type. A cycle $\mathcal{Z}$ on X with $\mathbb{Q}$-coefficients is *quasi-integral* if the multiplicity of every irreducible component Z_i of $\mathcal{Z}$ becomes an integer after multiplying it with the inseparable discrepancy of $k(Z_i)/k(S_i)$. Here S_i denotes the image of Z_i in S.

Theorem (8.6). *Let $S = \operatorname{Spec}(k)$ be the spectrum of a field. Then there is a one-to-one correspondence between relative cycles on X/S and effective cycles on X with quasi-integral coefficients. This correspondence is given by associating the underlying cycle to a relative cycle.*

Proof. It is clear from [Ryd08b, Prop. 8.6] that every cycle comes from at most one relative cycle. If α is a family on X/S then α has quasi-integral coefficients. In fact, let Z be an irreducible component of $\operatorname{Supp}(\alpha)$ and let e_Z be the exponent of $K(Z)/k$. Then $K(Z)^{e_Z}/k$ is separable and there is a separating transcendence basis $t_1, t_2, \dots, t_r$. The homomorphism $k[t_1, t_2, \dots, t_r] \to K(Z)^{e_Z} \to K(Z)$ extends to a morphism $U \to \mathbb{A}^r_k$ for some open subset $U \subseteq Z$. The inseparable discrepancy of $K(Z)/k$ coincides with the inseparable discrepancy of $K(Z)/K(t_1, t_2, \dots, t_r)$ and thus it follows from [Ryd08b, Prop. 8.11] that the multiplicity of α at Z is quasi-integral.

Conversely, let us show that the quasi-integral cycle $\frac{1}{e_Z}[Z]$ is the underlying cycle of a relative cycle. We can assume that $X = Z$. Let $(U, B, \mathbb{A}^n_k, p, g)$ be a smooth projection adapted to X. We want to construct a canonical relative zero-cycle $\alpha_{U/B/\mathbb{A}^n_k}$ on U/B with underlying cycle $\frac{1}{e_Z}[U]$. As B is normal (even regular), it is enough to construct this canonical relative zero-cycle over a generic point of B by Theorem (6.1). We can thus assume that U and B are irreducible. The inseparable discrepancy of $k(U)/k(B)$ is a multiple of the inseparable discrepancy of $k(X)/k$ and the existence of the relative cycle follows from [Ryd08b, Prop. 8.11]. $\square$

Corollary (8.7). *Let S be a reduced scheme. Then there is an injective map*

$$\mathit{Cycl}(X/S) \to \{\textit{quasi-integral effective cycles on } X\}$$

taking a relative cycle α on X/S to its underlying cycle.

Corollary (8.8). *Let S be a* reduced *scheme and let α be a relative cycle on X/S with support Z. Then α satisfies condition $(*)$ of Section 5 and α is the push-forward of a relative cycle on Z/S.*

Proof. If (U, B, T, p) is a smooth projection, then $p^* \operatorname{cycl}(\alpha) = \operatorname{cycl}(\alpha_{U/B/T})$. This shows condition $(*)$, i.e., that $\alpha_{U/B/T}$ does not depend on the morphisms $B \to T$ and $T \to S$. The last statement follows from Proposition (5.5). $\square$

Lemma (8.9). *Let $S = \operatorname{Spec}(k)$ be the spectrum of a field and let $\mathcal{Z}$ be an effective cycle with $\mathbb{Q}$-coefficients on X/S. Then $\mathcal{Z}$ is quasi-integral if and only if k is the intersection of all inseparable field extensions k'/k such that $\mathcal{Z}_{k'}$ has integral coefficients.*

Proposition (8.10). *Let X/S be a quasi-projective scheme with a given embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is a locally free $\mathcal{O}_S$-sheaf. Then there is functorial bijection between k-points of* $\operatorname{Chow}(X/S)$ *and k-points of* $\operatorname{ChowVar}(X \hookrightarrow \mathbb{P}(\mathcal{E}))$.

Proof. This follows from Lemma (8.9) and [Kol96, Thm. 4.5]. $\square$

Definition (8.11). Let α be a relative cycle on $f : X \to S$. We say that α is *multiplicity-free* at a point $x \in X$ if the geometric multiplicity of $\alpha_{f(x)}$ is one at the generic points of the irreducible components of $\operatorname{Supp}(\alpha)_{f(x)}$ containing x. We say that α is *normal* (resp. *smooth*) at x if α is multiplicity-free and equidimensional at x and $\operatorname{Supp}(\alpha_{f(x)}) = \big(\operatorname{Supp}(\alpha)_{f(x)}\big)_{\mathrm{red}}$ is geometrically normal (resp. smooth) at x over $k(f(x))$.

The requirement that α is equidimensional at x is explained by the following example:

Example (8.12). Let $S = \operatorname{Spec}(k[t])$ and $X = \operatorname{Spec}(k[t,x,y]/x(y,x-t))$. Then X is the union of a plane and a line meeting in the origin. The natural morphism $X \to S$ is smooth outside the origin. The special fiber X_0 is an affine line with an embedded point. The corresponding relative cycle $\mathcal{N}_{X/S}$ has underlying cycle $[X]$ and special fiber $[X_0] = [(X_0)_{\mathrm{red}}]$ which is smooth.

Note that the fact that α is multiplicity-free at $x \in X$, does *not* imply that $\operatorname{Supp}(\alpha)$ is reduced at x in its fiber. However, if $f : Z \to S$ is flat and $\alpha = \mathcal{N}_{Z/S}$, then α is multiplicity-free (resp. normal, resp. smooth) at $z \in Z$ if and only if Z is geometrically (R_0) (resp. geometrically normal, resp. smooth) at z in $Z_{f(z)}$.

9. Representable relative cycles

We have showed that if S is reduced, then any relative cycle on X/S is represented by an ordinary cycle on X/S, cf. Corollary (8.7). In this section, we will show that smooth relative cycles correspond to subschemes which are smooth over S and that if X/S is smooth then relative Weil divisors on X/S correspond to relative Cartier divisors on X/S. Unfortunately, these result are so far only proven when either S is reduced or S is of characteristic zero. I conjecture that these results hold in general.

It then follows (assuming that S is reduced or that S has characteristic zero), that multiplicity-free relative cycles and relative Weil divisors on (R_1)-schemes are represented by unique subschemes which are flat in relative codimension zero. When α is a relative cycle on X/S such that either S is reduced, α is multiplicity-free or α is a relative Weil divisor on a (R_1)-scheme (cases (A1)–(A3) in the introduction), α has several nice properties. In the following sections, these three cases are discussed in more detail.

Proposition (9.1). *Let $f : X \to S$ be an algebraic space, locally of finite type and let α be a relative cycle on X/S with support Z. The set of points $z \in Z$ such that α is multiplicity-free at z is open.*

Proof. Let $z \in Z$ be a point at which α is multiplicity-free and let $s = f(z)$. After replacing X with an open neighborhood of z, we can assume that every irreducible component of Z_s contains z. It is enough to show that α_r is multiplicity-free in a neighborhood of z for every r. Thus we can assume that Z is equidimensional of dimension r but it is now possible that $z \notin Z$.

After restricting X and S further, we can assume that there is an embedding $X \hookrightarrow \mathbb{A}^n_S$. There is then a projection $\pi_s : \mathbb{A}^n_s \to \mathbb{A}^r_s$ such that $\pi_s|_{Z_s}$ is quasi-finite in a neighborhood of z and generically étale. After restricting S, we can assume that this projection extends to a projection $\pi : \mathbb{A}^n_S \to \mathbb{A}^r_S$.

Choose étale morphisms $B \to \mathbb{A}^r_S$ and $U \to \mathbb{A}^n_S$ such that (U,B,S,p,g) is a projection adapted to Z with $z \in p(U)$. Then $\alpha_{U/B/S}$ is non-degenerate at the generic points of B_s. The non-degeneracy locus B_{nondeg} of the proper relative zero-cycle $\alpha_{U/B/S}$ is open. As the fibers of $B \to S$ are irreducible, it follows that α is multiplicity-free over the image of B_{nondeg} in S. This is an open subset as $B \to S$ is open. $\square$

Proposition (9.2). *Let $f : X \to S$ be an algebraic space, locally of finite type and let α be a relative cycle on X/S with support Z_0. Let $x \in X$ be a smooth point of α and let $s = f(x)$. Then there is a smooth projection (U,B,S,p), and a point $u \in U$ over x such that $p^{-1}((Z_0)_s)_{\text{red}} \to B_s$ is étale at x. If (U,B,S,p) is any such projection with $u \in U$ above x then, in a neighborhood of U, there exists a closed subscheme $Z \hookrightarrow U$ which is smooth over S and étale over B such that $\alpha_{U/B/S} = \mathcal{N}_{Z/B}$. In particular, the set of points $x \in X$ such that α is smooth at x is open.*

Proof. Let $x \in \operatorname{Supp}(\alpha)$ be a point at which α is smooth and let s be its image in S. Let $Z_0 = \operatorname{Supp}(\alpha) \hookrightarrow X$. Then $((Z_0)_s)_{\text{red}} \to \operatorname{Spec}(k(s))$ is smooth at x. Thus in a neighborhood of x there is a factorization $((Z_0)_s)_{\text{red}} \to \mathbb{A}^r_{k(s)} \to \operatorname{Spec}(k(s))$ such that $((Z_0)_s)_{\text{red}} \to \mathbb{A}^r_{k(s)}$ is étale at x. As $Z_0 \to S$ is equidimensional at x, this factorization lifts to a neighborhood $U \subseteq X$ of x such that $U \cap Z_0 \to \mathbb{A}^r_S$ is quasi-finite and dominant. We thus have a quasi-adapted projection and after étale localization, we obtain an adapted projection.

Now let (U,B,S,p) be any smooth projection such that $p^{-1}((Z_0)_s)_{\text{red}}$ is étale over B_s. Then passing to the fiber at s we obtain a family $\alpha_{U_s/B_s/s}$ which is étale, i.e., non-degenerate, at u. Thus, so is $\alpha_{U/B/S}$ in a neighborhood of u and the proposition follows with $Z = \operatorname{Image}(\alpha_{U/B/S})$. $\square$

Corollary (9.3). *Let $f : X \to S$ be an algebraic space, locally of finite type and let α be a relative cycle on X/S. Then* $\operatorname{Supp}(\alpha) \to S_{\text{red}}$ *is smooth at smooth points of α.*

Proof. We can assume that S is reduced. Then for any smooth projection (U,B,S) we have that B is reduced. It follows that $\operatorname{Image}(\alpha_{U/B/S}) = \operatorname{Supp}(\alpha_{U/B/S})$ and hence $\operatorname{Supp}(\alpha) \to S_{\text{red}}$ is smooth by Proposition (9.2). $\square$

Proposition (9.2) states that locally at $x \in X$ there is a *subscheme* $Z \hookrightarrow X$ such that $Z \to S$ is smooth and $\mathcal{N}_Z$ is equal to α under a certain projection. However, it does *not* follow trivially that this subscheme is independent on the choice of projection, except when S is reduced. At the moment, I can only show that Z is independent on the choice of projection in characteristic zero. We begin with two lemmas valid in arbitrary characteristic.

Lemma (9.4). *Let S be a scheme, let $X \to S$ be flat and locally of finite presentation and let $G \to S$ be proper and smooth with geometrically connected fibers. Let $S_0 = S_{\text{red}}$, $G_0 = G \times_S S_0$ and $X_0 = X \times_S S_0$. Let $Z_0 \hookrightarrow X_0$ be a subscheme which is flat and locally of finite presentation over S_0. Let $W_0 = Z_0 \times_{S_0} G_0$ and let $W \hookrightarrow X \times_S G$ be a subscheme such that $W \times_S S_0 = W_0$. Further, assume that $W \to G$ is flat and finitely presented* *over a schematically dense open retro-compact subset $U \subseteq W$ which contains all points of relative codimension one over Z_0. Then there exists a unique subscheme $Z \hookrightarrow X$, flat and locally of finite presentation over S, such that $Z_0 = Z \times_S S_0$ and $W = Z \times_S G$.*

Note that *a priori* $W \to G$ is only flat over U but that *a posteriori* it follows that $W \to G$ is flat. The lemma thus essentially states that all deformations of W_0 come from deformations of Z_0.

Proof. The question is local on X and S and we can thus assume that X and S are affine and that W and Z_0 are closed subschemes. By a limit argument, we can also assume that S is noetherian. By effective descent of closed subschemes for the smooth morphism $X \times_S G \to X$, the existence of a Z such that $W = Z \times_S G$ is equivalent to the condition that $\pi_1^{-1}W = \pi_2^{-1}W$ where π_1 and π_2 are the two projections $X \times_S G \times_S G \to X \times_S G$. This can be checked on infinitesimal neighborhoods of depth zero points on W, and hence on infinitesimal neighborhoods of depth zero points on S. We can thus assume that S is the spectrum of a local artinian ring A with maximal ideal $\mathfrak{m}$ and residue field k.

We will show the lemma by induction on the integer n such that $\mathfrak{m}^n = 0$. If $n = 1$, then there is nothing to prove. If $n > 1$, let $A_1 = A/\mathfrak{m}^{n-1}$ and let $J = \ker(A \to A_1)$ so that $J\mathfrak{m} = 0$. Then J is a k-module. Let $S_1 = \operatorname{Spec}(A_1)$ and let $X_1 = X \times_S S_1$, $G_1 = G \times_S S_1$ and $W_1 = W \times_S S_1$. Then by induction there is a subscheme $Z_1 \hookrightarrow X_1$, flat and finitely presented over S_1, such that $W_1 = Z_1 \times_{S_1} G_1$. Let $\mathcal{I}_i$ be the ideal sheaves defining $Z_i \hookrightarrow X_i$, for $i = 0, 1$, and let $p : U \times_G G_1 \to Z_1$ be the composition of the open immersion $j : U \times_G G_1 \to W_1$ and the projection $\pi : W_1 = Z_1 \times_{S_1} G_1 \to Z_1$.

By the deformation theory of Hilbert schemes, cf. [FGA, No. 221, p. 21] or [Kol96, I.2], the obstruction to extend the flat family $W_1|_U = p^{-1}Z_1$ over S_1 to a flat family over S is an element

$$c_J(W_1|_U) \in \operatorname{Ext}^1_{p^{-1}X_1}\bigl(p^*\mathcal{I}_1, \mathcal{O}_{W_0|_U} \otimes_k J\bigr) = \operatorname{Ext}^1_{X_1}\bigl(\mathcal{I}_1, p_*p^*(\mathcal{O}_{Z_0}) \otimes_k J\bigr).$$

As such an extension exists, namely the deformation $W|_U \to S$, the obstruction $c_J(W_1|_U)$ is zero. Moreover, $W|_U$ corresponds (non-canonically) to an element in

$$\operatorname{Hom}_{p^{-1}X_0}\bigl(p^*\mathcal{I}_0, \mathcal{O}_{W_0|_U} \otimes_k J\bigr) = \operatorname{Hom}_{X_0}\bigl(\mathcal{I}_0, p_*p^*(\mathcal{O}_{Z_0}) \otimes_k J\bigr).$$

Now, as $G \to S$ is proper and smooth, we have that π is cohomologically flat in dimension zero [EGA$_{\text{III}}$, Prop. 7.8.6] and as $G \to S$ has geometrically connected fibers it thus follows that $\pi_*\mathcal{O}_{W_0} = \mathcal{O}_{Z_0}$. As the open immersion j contains all points of depth one of W_0, it follows that $j_*j^*\mathcal{O}_{W_0} = \mathcal{O}_{W_0}$ and hence $p_*p^*\mathcal{O}_{Z_0} = \mathcal{O}_{Z_0}$. It follows that the obstruction

$$c_J(Z_1) \in \operatorname{Ext}^1_{X_1}\bigl(\mathcal{I}_1, \mathcal{O}_{Z_0} \otimes_k J\bigr)$$

is zero and that the deformation $W \to S$ of $W_1 \to S_1$ is the pull-back of a deformation $Z \to S$ of $Z_1 \to S_1$. □

We the need the following construction of Angéniol and El Zein [AEZ78, §I].

(9.5) *Grassmannians of projections* — Let S be a scheme, let $\mathcal{E} = \mathcal{O}_S^n$ be a free sheaf of rank n and let $X = \mathbb{A}^n_S = \operatorname{Spec}_S(\mathcal{E}^\vee)$. Let $\mathbb{G} = \mathbb{G}(r, n) = \mathbb{G}_r(\mathcal{E})$

be the grassmannian parameterizing quotients $\mathcal{E} \twoheadrightarrow \mathcal{F}$ such that $\mathcal{F}$ is locally free of rank r [$\mathrm{EGA}_{\mathrm{I}}$, 9.7]. Let $\pi : \mathbb{G} \to S$ be the structure morphism and let $\pi^*\mathcal{E} \twoheadrightarrow \mathcal{F}$ be the universal quotient. We then let $\mathbb{B} = \mathrm{Spec}_{\mathbb{G}}(\mathcal{F}^\vee)$. The morphism $\mathbb{B} \to \mathbb{G}$ is a vector bundle of rank r. The morphism $\mathbb{A}^n_{\mathbb{G}} = \mathrm{Spec}_{\mathbb{G}}(\mathcal{E}^\vee) \to \mathrm{Spec}_{\mathbb{G}}(\mathcal{F}^\vee) = \mathbb{B}$ is the *universal projection.*

Let $Z \hookrightarrow \mathbb{A}^n_S$ be a closed subset, equidimensional of dimension r over S. Let $U \subseteq Z \times_S \mathbb{G}$ be the open subset over which $Z \times_S \mathbb{G} \hookrightarrow \mathbb{A}^n_{\mathbb{G}} \to \mathbb{B}$ is quasi-finite. We say that $Z \hookrightarrow \mathbb{A}^n_S$ has property (P$'$) if $U \subseteq Z \times_S \mathbb{G}$ contains all points of relative codimension at most one over Z.

Lemma (9.6). *Let S be an affine scheme, let $X \to S$ be a scheme with a closed immersion $X \hookrightarrow \mathbb{A}^n_S$ and let $Z_0 \hookrightarrow X \times_S S_{\mathrm{red}}$ be a closed subscheme such that $Z_0 \to S_{\mathrm{red}}$ is a finitely presented morphism. Then there exists, Zariski-locally on X, a closed immersion $X \hookrightarrow \mathbb{A}^{n+m}_S$ such that the projection onto the first n factors is the original embedding of X in $\mathbb{A}^n_S$ and such that $Z_0 \hookrightarrow \mathbb{A}^{n+m}_S$ has property* (P$'$).

Proof. By a limit argument, there exists a noetherian scheme S_α, an affine morphism $S \to S_\alpha$ and a morphism of finite type $Z_\alpha \to (S_\alpha)_{\mathrm{red}}$ such that $Z_0 \to S_{\mathrm{red}}$ is the pull-back of $Z_\alpha \to (S_\alpha)_{\mathrm{red}}$ along $S_{\mathrm{red}} \to (S_\alpha)_{\mathrm{red}}$. By [AEZ78, Lem. I.3], every point $z \in Z_\alpha$ admits an open neighborhood V_α and a closed immersion $V_\alpha \hookrightarrow \mathbb{A}^m_{S_\alpha}$ satisfying (P$'$). If $V = Z_0 \times_{Z_\alpha} V_\alpha$, then the corresponding immersion $V \hookrightarrow \mathbb{A}^m_S$ also satisfies (P$'$). After replacing X with an open neighborhood of z, we can lift this immersion to a morphism $X \to \mathbb{A}^m_S$. We thus obtain an immersion $X \hookrightarrow \mathbb{A}^{n+m}_S$. By *loc. cit.* the immersion $Z_0 \hookrightarrow X \hookrightarrow \mathbb{A}^{n+m}_S$ satisfies (P$'$). □

Proposition (9.7). *Let S be purely of* characteristic zero *and let $X \to S$ be locally of finite type. Let α be a smooth relative cycle on X/S. Let (X, B_1, S) and (X, B_2, S) be two projections quasi-adapted to* $\mathrm{Supp}(\alpha)$. *Assume that there exists a locally closed subscheme $Z \hookrightarrow X$, such that $Z \to B_1$ is étale and such that $\alpha_{X/B_1/S} = \mathcal{N}_{Z/B_1}$. Then $\alpha_{X/B_2/S} = \mathcal{N}_{Z/B_2}$.*

Proof. Let $Z_0 \hookrightarrow X \times_S S_{\mathrm{red}}$ be the support of α. This is smooth over S_{red} by Corollary (9.3). The question is local on X and S and can be checked at neighborhoods of the generic points of Z_0. Taking étale projection $B_i \to \mathbb{A}^r_S$, we can assume that $B_i = \mathbb{A}^r_S$. Locally on X there is then a closed immersion $X \hookrightarrow \mathbb{A}^n_S$ such that the two projections $(X, \mathbb{A}^r_S, S)$ lifts to linear projections $\mathbb{A}^n_S \to \mathbb{A}^r_S$. We then take a closed immersion $X \hookrightarrow \mathbb{A}^{n+m}_S$ as in Lemma (9.6). We thus have a grassmannian $\mathbb{G} \to S$ and a projection $(X \times_S \mathbb{G}, \mathbb{B}, \mathbb{G})$ which is quasi-adapted to Z_0 over an open subscheme $U \subseteq X \times_S \mathbb{G}$ containing all points of relative codimension at most one over X. Furthermore, the two projections $X \to \mathbb{A}^r_S$ that we started with, appear as two of the fibers of the grassmannian family.

As the family of one of these fibers is non-degenerate, it follows that $\alpha_{U/\mathbb{B}/\mathbb{G}}$ is generically non-degenerate. It follows that there exists a closed subscheme $W \hookrightarrow X \times_S \mathbb{G}$ and an open subset $V \subseteq U \subseteq X \times_S \mathbb{G}$ such that $W|_V \subseteq W$ is schematically dense and $W|_V \to \mathbb{B}$ is étale. Furthermore, as S is of *characteristic zero*, it follows that V contains all points lying over a generic point of Z_0. Indeed, any quasi-finite morphism between regular schemes is generically étale in characteristic zero.

Replacing X with an open neighborhood of any generic point of Z_0 we can thus assume that $W|_U$ is smooth over $\mathbb{G}$. It follows from Lemma (9.4) that $W = Z \times_S \mathbb{G}$ for a unique subscheme $Z \hookrightarrow X$ with support Z_0. $\square$

If S is noetherian and such that the residue field of every points of depth zero has characteristic zero, then Proposition (9.7) is still true, as can be seen from the proof of Lemma (9.4). I do not know if the proposition is false in positive characteristic.

Theorem (9.8). *Let S be a scheme purely of characteristic zero and let X/S be locally of finite type. Let α be a smooth relative cycle on X/S. Then there is a unique subscheme $Z \hookrightarrow X$ which is smooth over S such that $\alpha = \mathcal{N}_Z$.*

Proof. Let $x \in Z$ and let (U, B, S, p, g, φ) be a projection with a lifting $u \in U$ of x as in Proposition (9.2) such that $\alpha_{U/B} = \mathcal{N}_W$ for a subscheme $W \hookrightarrow U$ which is smooth over S. We apply Proposition (9.7) with $(U \times_X U, B, S)$ and the two morphisms $\varphi_i : U \times_X U \to B$ given by the compositions of the projections and the morphism $\varphi : U \to B$. By étale descent, it then follows that there exists a subscheme $Z \hookrightarrow X$ which is smooth over S such that $W = Z \times_X U$.

Now let (U', B', T', p') be an arbitrary projection. We will show that $\alpha_{U'/B'/T'} = \mathcal{N}_{p'^{-1}(Z)/B'}$, and it suffices to show this equality étale-locally on U'. This follows from Proposition (9.7) applied on the two projections $(U' \times_X U, B', T', p' \circ \pi_1)$ and $(U' \times_X U, B \times_T T', T', p \circ \pi_2)$, $\square$

Corollary (9.9). *Let S be a scheme purely of characteristic zero and let X/S be locally of finite type. Let α be a relative cycle on X/S which is multiplicity-free. There is a unique subscheme $Z \hookrightarrow X$ which has support* $\operatorname{Supp}(\alpha)$ *and a fiberwise dense open subset $U \subseteq Z$, containing all associated points, such that $U \to S$ is smooth and such that $\mathcal{N}_{U/S} = \alpha|_U$. Moreover α is uniquely determined by Z. If $S' \to S$ is an arbitrary morphism, then the unique subscheme corresponding to $\alpha \times_S S'$ is the closure of $U \times_S S'$ in $Z \times_S S'$.*

Proof. Let Z_0 be the support of α. By Proposition (9.2), the subset $U_0 \subseteq Z_0$ of points where α is smooth is open. As α is multiplicity-free, this subset contains all points which are generic in their fibers, i.e., $U_0 \subseteq Z_0$ is fiberwise dense. Let $V \subseteq X$ be any open subset restricting to U_0. It then follows from Theorem (9.8) that $\alpha|_V = \mathcal{N}_{Z_V}$ for a unique subscheme $Z_V \hookrightarrow V$ which is smooth over S. This extends uniquely to a locally closed subscheme $Z \hookrightarrow X$ such that $Z|_V = Z_V$ is schematically dense in Z. $\square$

Corollary (9.10). *Let S be a scheme purely of characteristic zero and let X/S be locally of finite presentation. The morphism* $\operatorname{Hilb}_r^{\mathrm{red}}(X/S) \to \operatorname{Chow}_r^{\mathrm{red}}(X/S)$ *from the Hilbert functor parameterizing equidimensional and reduced subschemes of dimension r to the Chow functor parameterizing equidimensional and multiplicity-free families of cycles of dimension r is a monomorphism.*

Remark **(9.11).** If X/S is quasi-projective, it is not difficult to show that the above morphism is an immersion when restricted to a component $\mathrm{Hilb}_P^{\mathrm{red}}(X/S)$ where P is a polynomial of degree r. This also follows from the representability of $\mathrm{Hilb}_r(X/S)$ and $\mathrm{Chow}_r(X/S)$ for a projective scheme X/S as it then follows that $\mathrm{Hilb}_r(X/S) \to \mathrm{Chow}_r(X/S)$ is proper.

Proposition (9.12). *Let $f : X \to S$ be an algebraic space and let α be a relative cycle on X/S. Let $x \in X$ be a point such that α is a relative Weil divisor at x and f is smooth at x. Then there is a projection (U, B, S, p, g) quasi-adapted to* $\mathrm{Supp}(\alpha)$*, such that $p^{-1}(x)$ is non-empty and such that $U \to B$ is smooth. Furthermore, for any such projection, we have that $\alpha_{U/B/S} = \mathcal{N}_{Z/B}$ for a unique subscheme $Z \hookrightarrow U$ flat over B.*

Proof. The existence of the projection follows from an argument similar as in Proposition (9.2). The existence of Z follows from (2.9) as U/B is a smooth curve. □

Corollary (9.13). *Let $f : X \to S$ be smooth and let α be a relative Weil divisor on X/S. Then* $\mathrm{Supp}(\alpha) \to S_{\mathrm{red}}$ *is flat.*

As before we would like to show that Z is independent upon the choice of smooth projection but this is only accomplished in characteristic zero.

Proposition (9.14). *Let S be a scheme purely of* characteristic zero. *Let X/S be smooth and let α be a relative Weil divisor on X/S. Let (X, B_1, S) and (X, B_2, S) be two projections quasi-adapted to* $\mathrm{Supp}(\alpha)$*. Assume that $X \to B_1$ is smooth, such that $\alpha_{X/B_1/S} = \mathcal{N}_{W/B_1}$ for a locally closed subscheme $W \hookrightarrow X$, flat over B_1. Then $\alpha_{X/B_2/S} = \mathcal{N}_{W/B_2}$.*

Proof. Similar as Proposition (9.7) using (9.12). □

Theorem (9.15). *Let S be a scheme purely of* characteristic zero. *Let X/S be smooth and let α be a relative Weil divisor on X/S. Then there is a unique subscheme $Z \hookrightarrow X$ which is flat with Cohen-Macaulay fibers over S such that $\alpha = \mathcal{N}_Z$, i.e., Z is a relative Cartier divisor.*

Proof. Follows from Proposition (9.14) exactly as Theorem (9.8) follows from Proposition (9.7). □

Corollary (9.16). *Let S be a scheme purely of* characteristic zero. *Let $X \to S$ be locally of finite type and smooth at points of relative codimension at most one, e.g., $X \to S$ flat with* (R_1)*-fibers. Let α be a relative Weil-divisor on X/S. Then there is a unique subscheme $Z \hookrightarrow X$ which has support* $\mathrm{Supp}(\alpha)$ *and a fiberwise dense open subset $U \subseteq Z$, containing all associated points, such that $U \to S$ is a relative Cartier divisor and such that $\mathcal{N}_{U/S} = \alpha|_U$. The relative Weil divisor α is uniquely determined by Z. If $S' \to S$ is an arbitrary morphism, then the unique subscheme corresponding to $\alpha \times_S S'$ is the closure of $U \times_S S'$ in $Z \times_S S'$.*

Proposition (9.17). *Let X/S be locally of finite type and let α be a relative cycle on X/S. Assume that one of the following conditions are satisfied:*

(i) *S is reduced.*
(ii) *α is multiplicity-free and S is of characteristic zero.*

(iii) *X/S is smooth in relative codimension one, α is a relative Weil divisor and S is of characteristic zero.*

Then there is a locally closed subscheme $Z \hookrightarrow X$, such that $|Z| = \operatorname{Supp}(\alpha)$ and such that α is the push-forward of a relative cycle on Z/S. The relative cycle α satisfies condition $(*)$ of Section 5.

Proof. These assertions follow from Corollaries (8.7), (9.9) and (9.16). In fact, let Z be the representing subscheme in the last two cases and the support of α in the first case. Then for any smooth projection (U, B, T, p), the relative cycle $\alpha_{U/B/T}$ is determined by $p^{-1}(Z)$ (resp. $p^{-1}\operatorname{cycl}(\alpha)$ in the first case) and hence do not depend on T. This is condition $(*)$. Proposition (5.5) shows that α is the push-forward of a cycle on Z/S. □

10. Families over reduced parameter schemes

Let X be locally of finite type over a *reduced* scheme S. We describe the subset of effective cycles with $\mathbb{Q}$-coefficients which corresponds to the set of relative cycles on X/S, cf. Corollary (8.7). When S is semi-normal and of characteristic zero, we obtain the descriptions of Kollár [Kol96] and Suslin-Voevodsky [SV00]. When S is semi-normal and of positive characteristic, then the description is slightly different as Kollár does not include cycles with quasi-integral coefficients. This is a minor difference though, as Kollár has characterized the quasi-integral cycles. Suslin and Voevodsky work either with integral coefficients or with arbitrary rational coefficients. We also show that the fibers of a relative cycle can be computed via Samuel multiplicities of its underlying cycle.

Theorem (10.1). *Let S be* normal *with a finite number of irreducible components. Then there is a one-to-one correspondence between relative cycles on X/S and effective cycles on X with quasi-integral coefficients and universal open support.*

Proof. This follows from Theorem (8.6) and Corollary (6.2). □

Corollary (10.2). *Let S be normal with a finite number of irreducible components. Then the commutative monoid $Cycl_r^{\mathrm{cl}}(X/S)$ of r-dimensional cycles with closed support is freely generated by cycles of the form $(1/p^{\delta})[Z]$ where Z is an irreducible and reduced closed subscheme of X which is equidimensional of dimension r over S, and δ is the inseparable discrepancy of $k(Z)/k(S)$.*

Definition (10.3). Let X/S be locally of finite type, let $f : S' \to S$ be a morphism and let $\mathcal{Z} = \sum_i m_i[Z_i]$ be a cycle on X such that every irreducible component Z_i dominates an irreducible component of S. The *pull-back* of $\mathcal{Z}$ along f is the cycle $f^*\mathcal{Z} = \mathcal{Z} \times_S S' = \sum_i m_i[f^{-1}(Z_i)_{\mathrm{dom}/S'}]$.

The pull-back of a relative cycle *does not* correspond to taking the pull-back of the underlying cycle. This is because the underlying cycle need not be flat. Also, the pull-back of a cycle is not functorial as we forget all non-dominating and embedded components.

Proposition (10.4). *Let S be reduced and let α be a relative cycle on X/S. Assume that* $\operatorname{cycl}(\alpha) = \mathcal{Z} = \sum_i m_i[Z_i]$ *where the Z_i's are subschemes of X,* flat *and finitely presented over S, but not necessarily reduced or irreducible, and the m_i's are rational numbers. Let S' be reduced and let $S' \to S$ be any morphism. Then*

$$\operatorname{cycl}(\alpha \times_S S') = \mathcal{Z} \times_S S' = \sum_i m_i [Z_i \times_S S'].$$

Proof. The question is local on X and S and thus we can assume that X and S are quasi-compact. Let q be an integer clearing the denominators of the m_i's. As addition of cycles commutes with pull-back it is enough to show that $\operatorname{cycl}(q\alpha) \times_S S' = \sum_i q m_i [Z_i \times_S S']$ and we can thus assume that the m_i's are integers. Then $\alpha = \sum_i m_i \mathcal{N}_{Z_i}$ and it follows that $\alpha \times_S S' = \sum_i m_i (\mathcal{N}_{Z_i} \times_S S') = \sum_i m_i \mathcal{N}_{Z_i \times_S S'}$. □

Corollary (10.5). *Let S be a smooth curve, i.e., a noetherian regular scheme of dimension one, or the spectrum of a valuation ring. Let α be a relative cycle on X/S. Then for any point $s \in S$ we have that* $\operatorname{cycl}(\alpha_s) = \operatorname{cycl}(\alpha)_s$.

Proof. Follows from the previous proposition as any irreducible and reduced subscheme Z of X dominating S is flat over S. In fact, S is a Prüfer scheme, i.e., every finitely generated ideal of $\mathcal{O}_S$ is locally free. □

Definition (10.6) ([SV00, 3.1.1])**.** Let S be a scheme, let k be a field and let $s : \operatorname{Spec}(k) \to S$ be a point. A fat point over s is a triple (s_0, s_1, V) where V is a valuation ring and $s_0 : \operatorname{Spec}(k) \to \operatorname{Spec}(V)$ and $s_1 : \operatorname{Spec}(V) \to S$ are morphisms such that

(i) $s = s_1 \circ s_0$.
(ii) The image of s_0 is the closed point of $\operatorname{Spec}(V)$.
(iii) The image under s_1 of the generic point of $\operatorname{Spec}(V)$ is a generic point of S.

Remark **(10.7).** For every point $s \in S$ and generization $\xi \in S_{\max}$, there is a field extension $k/k(s)$ and a fat point (s_0, s_1, V) over $s : \operatorname{Spec}(k) \to S$ such that the image of the generic point by s_1 is ξ [$\mathrm{EGA_{II}}$, Prop. 7.1.4]. If S is locally noetherian, then there is a fat point with V a discrete valuation ring [$\mathrm{EGA_{II}}$, Prop. 7.1.7].

Proposition (10.8). *Let S be reduced and let α be a relative cycle on X/S. Let $s : \operatorname{Spec}(k) \to S$ be a point of S and (s_0, s_1, V) a fat point over s. Then*

$$\operatorname{cycl}(s^*\alpha) = s_0^*\big(s_1^* \operatorname{cycl}(\alpha)\big).$$

Proof. As s_1 is flat over the generic point, it is clear that $\operatorname{cycl}(s_1^*\alpha) = s_1^* \operatorname{cycl}(\alpha)$. The result thus follows from Corollary (10.5). □

The pull-back $s_0^* s_1^*$ can be interpreted as taking the limit fiber over s along a general curve through s.

Definition (10.9). Let S be reduced, let X/S be an algebraic space, locally of finite type and let $\mathcal{Z}$ be a cycle on X. We say that $\mathcal{Z}$ satisfies the *limit cycle condition* if for every point $s : \operatorname{Spec}(k) \to S$, the pull-back $s_0^* s_1^* \mathcal{Z}$ is independent on the choice of fat point (s_0, s_1, V) over s. When $\mathcal{Z}$ satisfies the limit cycle condition, then we let $s^{[-1]}(\mathcal{Z})$ denote the pull-back $s_0^* s_1^* \mathcal{Z}$

for any choice of fat point over s, under the assumption that there exists a fat point over s.

Proposition (10.10). *Let S be reduced, let X/S be locally of finite type and let $\mathcal{Z}$ be a cycle on X flat over S, i.e., $\mathcal{Z} = \sum_i m_i[Z_i]$ where the Z_i's are flat over S. Then $\mathcal{Z}$ satisfies the limit cycle condition and $s^{[-1]}(\mathcal{Z}) = \mathcal{Z}_s$.*

Proof. Trivial, as the pull-back of a flat cycle is functorial. □

Corollary (10.11). *Let X/S be locally of finite type, and let $\mathcal{Z}$ be a cycle on X. Let $f : S' \to S$ be a proper morphism such that $\mathcal{Z}' = f^*\mathcal{Z}$ is flat over S'. Then $\mathcal{Z}$ satisfies the limit cycle condition if and only if for any point $s : \mathrm{Spec}(k) \to S$ the cycle $\mathcal{Z}'_{s'}$ is independent on the choice of a lifting $s' : \mathrm{Spec}(k) \to S'$ of s. If this is the case, then $s^{[-1]}\mathcal{Z} = \mathcal{Z}'_{s'}$ for any such lifting.*

Proof. Follows easily from the valuative criterion for proper morphisms and the previous proposition. □

If S is reduced and noetherian and X/S is of finite type, then there exists a proper morphism $S' \to S$ which flatifies $\mathcal{Z}$. In fact, under these hypotheses there is an open dense subset $U \subseteq S$ such that $\mathcal{Z}$ is flat over U [$\mathrm{EGA}_{\mathrm{IV}}$, Cor. 11.3.2]. If $\mathrm{Supp}(\mathcal{Z})$ is proper over S, the existence of $S' \to S$ then follows from the existence of the Hilbert scheme $\mathrm{Hilb}(\mathrm{Supp}(\mathcal{Z})/S)$. In the non-proper case, this is Raynaud and Gruson's flatification theorem [RG71].

Lemma (10.12). *Let S be reduced, let X/S be locally of finite type and let $\mathcal{Z}$ be a cycle on X satisfying the limit cycle condition and such that any component of $\mathcal{Z}$ dominates a component of S. Then for any point $s \in S$, the support of $s^{[-1]}\mathcal{Z}$ equals the support of* $\mathrm{Supp}(\mathcal{Z})_s$. *Also, the support of $\mathcal{Z}$ satisfies condition* (T) *universally.*

Proof. Let $Z = \mathrm{Supp}(\mathcal{Z})$. Let $z \in Z$ be a point and choose a generization $\eta \in Z_{\max}$. Let $s \in S$ and $\xi \in S_{\max}$ be the images of z and η. Choose a valuation ring V and a morphism $\mathrm{Spec}(V) \to X$ such that the closed point v_0 is mapped onto z and the generic point v_1 is mapped onto η. Let $s_1 : \mathrm{Spec}(V) \to S$ be the composition of $\mathrm{Spec}(V) \to X$ and $X \to S$. Let $k/k(v_0)$ be an extension such that k is algebraically closed and let $s_0 : \mathrm{Spec}(k) \to \mathrm{Spec}(k(v_0)) \hookrightarrow \mathrm{Spec}(V)$ be the corresponding morphism. Then $X \times_S \mathrm{Spec}(V) \to \mathrm{Spec}(V)$ has a section mapping v_0 onto the $k(v_0)$-point (z, v_0). It follows that (z, v_0) is in the support of $s_1^*\mathcal{Z}$ and hence that $(z, s_1 \circ s_0)$ is in the support of $s_0^* s_1^*\mathcal{Z}$. For any $\psi \in \mathrm{Aut}_{k(s)}(k)$ we have by assumption that $s_0^* s_1^*\mathcal{Z} = (s_0 \circ \psi)^* s_1^*\mathcal{Z}$. Thus any closed point in $(s_1 \circ s_0)^{-1}Z$ above z is contained in the support of $s_0^* s_1^*\mathcal{Z}$. It follows that $s_0^* s_1^*\mathcal{Z}$ contains the whole fiber above z. Thus $\mathrm{Supp}(s^{[-1]}\mathcal{Z}) = |Z_s|$.

In particular, $\mathrm{Supp}(s_1^*\mathcal{Z}) = |s_1^{-1}Z|$ for any valuation ring V and morphism $s_1 : \mathrm{Spec}(V) \to S$. It follows from Proposition (3.6) that Z/S satisfies (T) universally. □

We denote by $k^{\mathrm{perf}} = k^{p^{-\infty}}$ the perfect closure of k, where p is the characteristic of k.

Proposition (10.13). *Let S be reduced, let X/S be locally of finite type and let $\mathcal{Z}$ be a cycle on X satisfying the limit cycle condition and such that any component of $\mathcal{Z}$ dominates a component of S. For any point $s \in S$ there is a unique cycle $s^{[-1]}\mathcal{Z}$ on $X \times_S \operatorname{Spec}(k(s))$ such that for any field extension $k/k(s)$ and fat point (s_0, s_1, V) over $\operatorname{Spec}(k) \to \operatorname{Spec}(k(s)) \hookrightarrow S$, the cycle $(s_0^* s_1^* \mathcal{Z})$ coincides with $s^{[-1]}\mathcal{Z} \times_{k(s)} \operatorname{Spec}(k)$.*

Proof. From the previous lemma, it follows that the support of $s^{[-1]}\mathcal{Z}$ should be $\operatorname{Supp}(\mathcal{Z})_s = \operatorname{Supp}(\mathcal{Z}) \times_S \operatorname{Spec}(k(s)^{\mathrm{perf}})$. Thus, it is enough to assign multiplicities for the irreducible components of $\operatorname{Supp}(\mathcal{Z})_s$. If $W \subseteq \operatorname{Supp}(\mathcal{Z})_s$ is an irreducible component, then there is a finite separable and normal field extension $k/k(s)^{\mathrm{perf}}$ such that the irreducible components of W_k are geometrically irreducible [EGA$_{\mathrm{IV}}$, Cor. 4.5.11]. It then follows from the limit cycle condition, and the action of $Gal(k/k(s)^{\mathrm{perf}})$ on any algebraically closed extension of k, that the multiplicities of the irreducible components of W_k are all equal. The multiplicity of $s^{[-1]}\mathcal{Z}$ at W is then this common value divided by the inseparable degree of $k(W)/k(s)$. $\square$

Recall that a morphism $f : X \to Y$ is *integral* if f is affine and $f_*\mathcal{O}_X$ is integral over $\mathcal{O}_Y$. A morphism $f : X \to Y$ is a *universal homeomorphism* if $f' : X' \to Y'$ is a homeomorphism for any base change $Y' \to Y$. A morphism of *schemes* $f : X \to Y$ is a universal homeomorphism if and only if f is integral, universally injective and surjective [EGA$_{\mathrm{IV}}$, Cor. 18.12.11]. The same holds for a *locally separated* morphism of algebraic spaces [Ryd07, Cor. 4.22]. We recall the following definitions, cf. [AB69, Tra70, Swa80, Man80, Yan83, Kol96, Ryd07].

Definition (10.14). A morphism $f : X \to Y$ is weakly subintegral (resp. subintegral) if it is a separated universal homeomorphism (resp. a separated universal homeomorphism with trivial residue field extensions). A reduced algebraic space X is *weakly normal* (resp. semi-normal) if every birational weakly subintegral (resp. subintegral) morphism $X' \to X$, from a reduced space X', is an isomorphism.

Let $f : X \to Y$ be a morphism. Consider the set of factorizations $X \to Y' \to Y$ of f such that $X \to Y'$ is schematically dominant and $g : Y' \to Y$ is subintegral (resp. weakly subintegral). We have corresponding homomorphisms $\mathcal{O}_Y \to g_*\mathcal{O}_{Y'} \hookrightarrow f_*\mathcal{O}_X$ and as g is affine, the set of such factorizations is partially ordered with $Y_1' \geq Y_2'$ if and only if there exists a morphism $Y_1' \to Y_2'$ or equivalently if and only if $(g_2)_*\mathcal{O}_{Y_2'} \subseteq (g_1)_*\mathcal{O}_{Y_1'}$. The *subintegral closure*, or semi-normalization, $Y^{X/\mathrm{sn}} \to Y$ (resp. *weak subintegral closure* or weak normalization $Y^{X/\mathrm{wn}} \to Y$) of f is the maximal element in this set.

If X is an algebraic space with a finite number of irreducible components, then the semi-normalization X^{sn} (resp. weak normalization X^{wn}) is the subintegral closure (resp. weak subintegral closure) of X with respect to the normalization $\widetilde{X} \to X$. As (weakly) subintegral morphisms are integral, it follows that X^{sn} is semi-normal and that X^{wn} is weakly normal.

The following proposition is a special form of "h-descent". In general, if $S' \to S$ is universal subtrusive of finite presentation (e.g. faithfully flat or proper and surjective) and X is a scheme, then the sequence

$$\mathrm{Hom}(S,X) \longrightarrow \mathrm{Hom}(S',X) \rightrightarrows \mathrm{Hom}\big((S'\times_S S')_{\mathrm{red}},X\big)$$

is exact if S is absolutely weakly normal [Voe96, Ryd07].

Proposition (10.15). *Let S be a reduced scheme, X/S an algebraic space, locally of finite type and let $p : S' \to S$ be an integral surjective morphism of reduced schemes. Let $S'' = (S'\times_S S')_{\mathrm{red}}$ and denote the two projections by π_1 and π_2. Let α' be a relative cycle on X'/S' such that $\pi_1^*\alpha' = \pi_2^*\alpha'$. Assume that either of the following conditions is satisfied.*

(i) *S is weakly subintegrally closed in S',*
(ii) *S is subintegrally closed in S' and for any $s \in S$, there exists a relative cycle α_s on $X_s/\mathrm{Spec}(k(s))$ such that $\alpha_s \times_s p^{-1}(s) = \alpha' \times_{S'} p^{-1}(s)$.*

Then there exists a unique relative cycle α on X/S such that $\alpha' = p^*\alpha$.

Proof. Let $Z' = \mathrm{Supp}(\alpha') \hookrightarrow X'$ and let $Z \hookrightarrow X$ be the image of Z'. As $X' \to X$ is universally closed and $Z' = p^{-1}(Z)$, it follows that Z is a locally closed subset of X. As the support commutes with arbitrary base change, we have that $\pi_1^{-1}(Z') = \pi_2^{-1}(Z')$ and hence that $Z' = p^{-1}(Z)$. The support of α, if it exists, is Z.

As the (weak) subintegral closure and the reduction commutes with smooth base change [Ryd07, App. B] we can take a smooth projection adapted to Z and assume that $Z \to S$ is finite. Then α' corresponds to a morphism $\alpha' : S' \to \Gamma^\star(X/S)$ such that $\alpha'\circ\pi_1 = \alpha'\circ\pi_2$. Moreover, as S' is reduced, it follows that α' factors through $\Gamma^\star(Z'/S')$ and hence through $\Gamma^\star(Z/S)$. Note that $\Gamma^\star(Z/S)$ is finite, and in particular affine, over S.

Let W be the image of $\alpha' : S' \to \Gamma^\star(Z/S)$ and consider the factorization $S' \to W \to S$. As $\alpha\circ\pi_1 = \alpha'\circ\pi_2$ we obtain a bijective section $\alpha : S \to W$ of *sets* such that $\alpha' = \alpha\circ p$. As $S' \to S$ is *submersive*, i.e., S is equipped with the quotient topology, this section is *continuous* and it follows that $W \to S$ is *weakly subintegral*, i.e., a universal homeomorphism. If α'_s lifts to a morphism $\alpha_s : k(s) \to W$ for every $s \in S$, then $W \to S$ is *subintegral.* Thus $W = S$ under either of the two conditions and α' lifts to a *morphism* $\alpha : S = W \hookrightarrow \Gamma^\star(Z/S)$. □

Theorem (10.16). *Let S be* weakly normal *with a finite number of components. Then there is a one-to-one correspondence between relative cycles α on X/S and effective cycles $\mathcal{Z}$ on X such that:*

(i) *Every irreducible component of $\mathcal{Z}$ dominates an irreducible component of S.*
(ii) *$\mathcal{Z}$ satisfies the limit cycle condition.*
(iii) *$\mathcal{Z}$ has quasi-integral coefficients, i.e., for any* generic *point $s \in S_{\max}$, the cycle $\mathcal{Z}_s$ has quasi-integral coefficients.*

Proof. If α is a relative cycle then $\mathrm{cycl}(\alpha)$ satisfies the three conditions. Indeed, the first follows by definition, the second follows from Proposition (10.8) and the third from Theorem (8.6).

Conversely, assume that we are given a cycle $\mathcal{Z}$ satisfying the three conditions. Let $S' \to S$ be the normalization. Then by Theorem (10.1) we have that $\mathcal{Z} \times_S S' = \operatorname{cycl}(\alpha')$ for a unique relative cycle α' on X'/S'. Let $S'' = (S' \times_S S')_{\mathrm{red}}$ and denote the two projections with π_1 and π_2. Then $\pi_1^*\alpha' = \pi_2^*\alpha'$. In fact, for any point $s'' \in S''$ we have that $(\pi_1^*\alpha')_{s''} = (\pi_2^*\alpha')_{s''}$ as their underlying cycles coincide with $s''^{[-1]}\mathcal{Z}$. The theorem then follows by h-descent, cf. Proposition (10.15). □

Theorem (10.17). *Let S be* semi-normal *with a finite number of components. Then there is a one-to-one correspondence between relative cycles α on X/S and effective cycles $\mathcal{Z}$ on X such that:*

(i) *Every irreducible component of $\mathcal{Z}$ dominates an irreducible component of S.*
(ii) *$\mathcal{Z}$ satisfies the limit cycle condition.*
(iii) *For every $s \in S$, the cycle $s^{[-1]}\mathcal{Z}$ has quasi-integral coefficients.*

In particular, a relative cycle such that its underlying cycle has integral *coefficients, is a well defined family of cycles satisfying the Chow-field condition in the terminology of Kollár* [Kol96, Defs. I.3.10, I.4.7].

Proof. Reason as in the proof of Theorem (10.16). □

Corollary (10.18). *Let S be a* semi-normal *scheme over* $\operatorname{Spec}(\mathbb{Q})$ *with a finite number of components. Then there is a one-to-one correspondence between relative cycles α on X/S and effective cycles $\mathcal{Z}$ on X such that:*

(i) *Every irreducible component of $\mathcal{Z}$ dominates an irreducible component of S.*
(ii) *$\mathcal{Z}$ satisfies the limit cycle condition.*

In particular, under this hypothesis on S, a relative cycle corresponds to a relative effective cycle in the terminology of Suslin and Voevodsky [SV00, Def. 3.1.3].

Corollary (10.19) ([Bar75, Ch. II, §3])**.** *Let S be semi-normal and let $\mathcal{Z}$ be a cycle on X/S. Then there is a one-to-one correspondence between relative cycles α on X/S and effective cycles $\mathcal{Z}$ on X such that:*

(i) *The support of $\mathcal{Z}$ satisfies* (T).
(ii) *There is a smooth projection (U, B, S, p) such that* $\operatorname{Supp}(\alpha) \subseteq p(U)$ *and such that $p^*(\mathcal{Z})$ satisfies the limit cycle condition over B.*
(iii) *For every $s \in S$, the cycle $s^{[-1]}\mathcal{Z}$ has quasi-integral coefficients.*

Proof. This follows from the observation that the limit cycle condition on X/S is equivalent to the limit cycle condition on U/B. □

Finally, we define the pull-back of a relative cycle using intersection theory.

Definition (10.20) ([Ful98, Ex. 4.3.4])**.** Let $W \hookrightarrow Z$ be a closed subscheme with irreducible components $\{W_i\}$. The multiplicity of Z along W at W_i, denoted $(e_W Z)_{W_i}$, is the Samuel multiplicity of the primary ideal determined by W in the local ring $\mathcal{O}_{Z,w_i}$ where w_i is a generic point of W_i.

If $[Z] = \sum_j m_j[Z_j]$ then $(e_W Z)_{W_i} = \sum_j m_j (e_{W \cap Z_j} Z_j)_{W_i}$ [Ful98, Lem. 4.2]. This motivates the following definition:

Definition (10.21). Let S be reduced with a finite number of irreducible components and let X/S be locally of finite type. Let $Z \hookrightarrow X$ be an irreducible locally closed subscheme and let $s \in S$. We denote by $[Z]_s$ the cycle

$$\sum_V \frac{(e_{Z_s}Z)_V}{e_s S}[V]$$

where the sum is taken over the irreducible components of Z_s. We extend this definition linearly to cycles on X.

We have the following generalization of [SV00, Thm. 3.5.8]:

Theorem (10.22). *Let S be a reduced scheme with a finite number of irreducible components and let α be a relative cycle on $f : X \to S$ with underlying cycle $\mathcal{Z} = \mathrm{cycl}(\alpha)$. Then for any point $s \in S$ we have that $\mathrm{cycl}(\alpha_s) = [\mathrm{cycl}(\alpha)]_s$.*

Proof. Let $Z = \mathrm{Supp}(\alpha)$. Let $V \hookrightarrow Z_s$ be an irreducible component with generic point v. Let (U, B, S, p) be a smooth projection adapted to Z such that there exists a point $v' \in U$ above v such that v' is the only point of $p^{-1}(Z)$ in its fiber over B. Let $V' \hookrightarrow p^{-1}(Z_s)$ be the corresponding irreducible component. Let $W \hookrightarrow B_s$ be the image of V' — this is a connected component of B_s — and let w be its generic point.

Then since $p : U \to X$ and $B \to S$ are smooth it follows from [SV00, Lem. 3.5.2] that $e_W B = e_s S$ and

$$\big(e_{p^{-1}(Z_s)} p^* \mathcal{Z}\big)|_{V'} = (e_{Z_s}\mathcal{Z})_V.$$

Thus, if we show that $e_{p^{-1}(Z_s)}(p^*\mathcal{Z})|_{V'}/e_W B$ is the multiplicity of $\alpha_{U/B/S}$ at v', the result follows. Replacing X, S and s with U, B and w, we can thus assume that α is a proper relative zero-cycle such that Z_s consists of a single (non-reduced) point z.

Let S_j be an irreducible component of S and let $\mathcal{Z}_j = \mathcal{Z}|_{S_j} = \sum_i m_i[Z_i]$ be the pull-back of the cycle to S_j. Then

$$e_{(Z_i)_s} Z_i = \frac{e_s f_*[Z_i]}{\deg\big(k(z)/k(s)\big)} = e_s S_j \frac{\deg\big(k(Z_i)/k(S_j)\big)}{\deg\big(k(z)/k(s)\big)},$$

cf. [SV00, Lem. 3.5.3]. Thus

$$e_{Z_s}\mathcal{Z}_j = e_s S_j \frac{\deg(\alpha)}{\deg\big(k(z)/k(s)\big)} = e_s S_j \, \mathrm{mult}_z(\alpha)$$

and the theorem follows. □

Corollary (10.23). *Let S be a smooth scheme and let α be an equidimensional relative cycle on X/S. Then the pull-back of α coincides with the pull-back of* $\mathrm{cycl}(\alpha)$ *given by intersection theory. That is,* $\mathrm{cycl}(\alpha_s) = \mathrm{cycl}(\alpha)_s$ *where the right-hand side is the cycle (not the rational equivalence class) defined in* [Ful98, §10.1].

Proof. As S is smooth, $e_s S = 1$, and thus the corollary follows from the theorem and [Ful98, 10.1.1]. □

11. Multiplicity-free relative cycles and relative Weil divisors

In Section 9 we saw that multiplicity-free relative cycles and relative Weil divisors on (R_1)-schemes are given by unique subschemes which are flat over a fiberwise dense open subset. Conversely, we would like to characterize the subschemes, fiberwise generically flat, which correspond to such relative cycles. This is not accomplished in general. We only mention the simple cases in which such correspondences are known.

Note that under these correspondences, the pull-back of a relative cycle corresponds to the ordinary pull-back of the corresponding subscheme after removing embedded components of relative codimension at least one.

Let X be a locally noetherian scheme. We recall that X is ($\mathrm{R_n}$) if X is regular at every point of codimenson n and ($\mathrm{S_n}$) if every point of depth $d \leq n$ has codimension d. In particular, X is (R_0) if it is reduced at every generic point and (S_1) if it has no embedded components. A scheme is (S_2) if it is (S_1) and every point of depth 1 has codimension 1. Serre's condition states that X is normal if and only if X is (R_1) and (S_2).

Recall that a morphism f, locally of finite type, is reduced (resp. normal, resp. ($\mathrm{R_n}$), resp. ($\mathrm{S_n}$)) if it is flat and its geometric fibers are reduced (resp. normal, etc.) [$\mathrm{EGA_{IV}}$, Def. 6.8.1]. We say that a relative cycle α on X/S is ($\mathrm{R_n}$), if α_s is multiplicity-free and $\mathrm{Supp}(\alpha_s)$ is geometrically ($\mathrm{R_n}$) for every $s \in S$.

Definition (11.1). Let $Z \to S$ be locally of finite type. We say that $Z \to S$ is *n-flat* (resp. *n-smooth*) if there exists a schematically dense open subset $U \subseteq Z$, containing all points of relative codimension at most n, such that $U \to S$ is flat, (S_1) and locally of finite presentation (resp. smooth).

The condition that U is schematically dense is equivalent to demanding that all (weakly) associated points of Z have relative codimension zero. Indeed, by flatness and the (S_1)-condition, any associated point in U has relative codimension zero.

Remark **(11.2).** If $Z \to S$ is 0-flat, then $Z \to S$ satisfies the condition (T) universally, i.e., $Z' \to S'$ satisfies (T) after any base change $S' \to S$. In particular, if $Z \to S$ is 0-flat and equidimensional, then $Z \to S$ is universally equidimensional. If in addition $Z \to S$ is locally of finite presentation or S has a finite number of components, then $Z \to S$ is universally open. This follows from [$\mathrm{EGA_{IV}}$, Cor. 1.10.14] and Corollary (6.3).

Conceptually, an n-flat morphism is a family of (S_1)-schemes, i.e., schemes without embedded components. Of course, the ordinary fibers are not necessarily (S_1)-schemes but this is taken care of by the following definition

Definition (11.3). If $g : S' \to S$ is a morphism and $Z \to S$ is 0-flat, then we let $g^*_{\mathrm{emb}}(Z)$ be the closure of $U \times_S S'$ in $Z \times_S S'$ for some $U \subseteq Z$ as in the definition of 0-flat.

Note that since $U \times_S S' \subseteq Z \times_S S'$ is dense, $g^*_{\mathrm{emb}}(Z)$ has the same support as the usual pull-back. Also, $g^*_{\mathrm{emb}}(Z)$ can be described as removing all embedded components of relative codimension at least one. In particular,

$g^*_{\mathrm{emb}}(Z)$ does not depend upon the choice of U. If Z is n-flat (resp. n-smooth) then $g^*_{\mathrm{emb}}(Z)$ is n-flat (resp. n-smooth).

Remark **(11.4).** Let X/S be locally of finite presentation. If $Z \hookrightarrow X$ is a subspace and $Z \to S$ is 0-flat with $U \subseteq Z$ as in the definition of 0-flat, then Z/S defines a relative cycle $\mathcal{N}_{U/S}$ on X/S. By Corollary (6.6), this relative cycle has at most one extension to Z. If $Z \to S$ is 1-flat or S is reduced, then the same corollary (together with a limit argument in the non-noetherian case) shows that such an extension exists. We will denote this extension by $\mathcal{N}_{Z/S}$.

Theorem (11.5). *Let X/S be locally of finite type. There is a one-to-one correspondence between multiplicity-free relative cycles on X/S and subschemes $Z \hookrightarrow X$ such that $Z \to S$ is* 0*-smooth and $\mathcal{N}_U$ extends to a cycle on Z. Under this correspondence, the pull-back of a relative cycle corresponds to the pull-back of a* 0*-smooth morphism as defined in Definition* (11.3)*. In particular, we have the following correspondences:*

(i) *If S is* reduced*, there is a one-to-one correspondence between multiplicity-free relative cycles on X/S and subschemes $Z \hookrightarrow X$ such that Z/S is* 0*-smooth.*
(ii) *For arbitrary S, and $n \geq 1$, there is a one-to-one correspondence between relative* $(\mathrm{R_n})$*-cycles on X/S and subschemes $Z \hookrightarrow X$ such that $Z \to S$ is n-smooth.*

Proof. As 0-smooth morphisms satisfies condition (T), the first correspondence follows from Corollary (9.9). The last two correspondences follows from Remark (11.4) and Theorem (9.8). □

Let X/S be flat and locally of finite presentation. An effective *relative Cartier divisor* on X/S is an immersion $Z \hookrightarrow X$ which is transversally regular relative to S of codimension one [$\mathrm{EGA_{IV}}$, 21.15.3.3]. By definition, this means that Z/S is flat and that $Z \hookrightarrow X$ is a Cartier divisor. Equivalently, Z/S is flat and $Z_s \hookrightarrow X_s$ is a Cartier divisor for every $s \in S$ [$\mathrm{EGA_{IV}}$, Prop. 19.2.4].

Definition (11.6). Let X/S be $(n+1)$-flat. We say that a subscheme $Z \hookrightarrow X$ is n-Cartier if $Z|_U \hookrightarrow X|_U$ is a relative Cartier divisor for some open subset $U \subseteq X$ containing all point of relative codimension $n+1$.

By definition, if $Z \hookrightarrow X$ is n-Cartier, then Z/S is n-flat. An n-flat subscheme $Z \hookrightarrow X$ is n-Cartier if and only if $Z_s \hookrightarrow X_s$ is n-Cartier.

Theorem (11.7). *Let X/S be locally of finite type and* 1*-smooth. There is a one-to-one correspondence between relative Weil divisors on X/S and subschemes $Z \hookrightarrow X$ which are* 0*-Cartier and such that $\mathcal{N}_U$ extends to Z. The pull-back of relative Weil divisor corresponds to the pull-back of* 0*-flat morphisms. We also have the following correspondences.*

(i) *If S is* reduced*, then there is a one-to-one correspondence between relative Weil-divisors on X/S and subschemes $Z \hookrightarrow X$ which are* 0*-Cartier.*

(ii) *If X/S is $(n+1)$-smooth, for some $n \geq 1$, then there is a one-to-one correspondence between relative Weil divisors on X/S and subschemes $Z \hookrightarrow X$ which are n-Cartier.*

Proof. Follows from Theorem (9.15) as in the proof of Theorem (11.5). □

Corollary (11.8). *Let X/S be smooth of dimension $r+1$. Then* $\mathrm{Chow}_r(X/S)$ *is isomorphic to the functor* $\mathrm{Div}(X/S)$ *parameterizing relative Cartier divisors on X/S.*

When X/S is smooth of relative dimension $r+1$, then the morphism

$$\mathrm{Hilb}_{r-1}(X/S) \to \mathrm{Chow}_{r-1}(X/S) \cong \mathrm{Div}(X/S),$$

taking a proper family of subschemes of dimension $r-1$ to the corresponding equidimensional relative cycle, can be described as follows. Let $\mathcal{F}$ be a quasi-coherent sheaf on X with support of dimension $r-1$ such that $\mathcal{F}$ is flat over S. Then $\mathcal{F}$ has finite Tor-dimension over X by Lemma (7.12). The *determinant* of $\mathcal{F}$, denoted $\det(\mathcal{F})$ is the alternating determinant of a locally free resolution of $\mathcal{F}$ [GIT, Ch. 5, §3], [Fog69, §2], [KM76]. This is a locally free sheaf on X and there is a section of $\det(\mathcal{F})$ which is unique up to a unit in $\mathcal{O}_X$. This determines an effective Cartier divisor on X and the corresponding relative cycle coincides with $\mathcal{N}_{\mathcal{F}}$ by Proposition (7.13). The morphism $\mathrm{Hilb}_{r-1}(X/S) \to \mathrm{Div}(X/S)$ was used by Fogarty to study the Hilbert scheme of a smooth surface [Fog68].

In [Fog71], Fogarty considers families of Weil divisors on a projective (R_1)-scheme X/k which is equidimensional of dimension r. He then defines a relative Weil-divisor on $X \times_k S/S$ as a subscheme $Z \hookrightarrow X \times S$ which is Cartier over the smooth locus of X. Thus, when either S is reduced or X/k is (R_2), Fogarty's definition agrees with our definition. Fogarty then shows [Fog71, Prop. 4.4] that the classical Chow construction, reviewed in Section 17, extends to give a morphism $\mathrm{Chow}_{r-1,d}(X) \to \mathrm{Div}_d(G)$ under one of the following conditions.

(i) S is normal.
(ii) X/k is (R_2) and (S_2).

The results of Section 17 shows that such a morphism exists if either S is reduced or X/k is (R_2). Conjecturally, this morphism exists without any assumptions on S and X, but then the elements of $\mathrm{Chow}_{r-1}(X)(S)$ are not represented by subschemes of $X \times S$.

Fogarty also shows [Fog71, §5], assuming that S is reduced or X/k is (R_2), that the morphism $\mathrm{Chow}_{r-1,d}(X)(S) \to \mathrm{Div}_d(G)(S)$ is injective. Finally [Fog71, §6] he shows that the normalization of $\mathrm{Chow}_{r-1,d}(X)$ is representable (this is simply the normalization of the classical Chow variety) and that if X/k is (R_2), then $\mathrm{Chow}_{r-1,d}(X)_{\mathrm{red}}$ is representable (i.e., the classical Chow variety is independent of the embedding in this case).

12. Relative normal cycles

In this section we prove a generalized version of Hironaka's lemma. The standard version of Hironaka's lemma is that if S is the spectrum of a discrete valuation ring and $X \to S$ is an equidimensional morphism such that the generic fiber is normal, the special fiber is generically reduced and the reduction of the special fiber is normal, then the special fiber is normal. In the terminology of the previous section, Grothendieck and Seydi's [GS71] generalization of Hironaka's lemma states that if S is *reduced* and $X \to S$ is 0-smooth and equidimensional and such that the reduction of any fiber is normal, then $X \to S$ is normal.

The version of Hironaka's lemma that we will prove states that for arbitrary S, any 1-smooth equidimensional morphism $X \to S$ such that the reduction of its fibers are normal, is normal.

Lemma (12.1). *Let S be a locally noetherian scheme and let X be a locally noetherian S-scheme. Let $X' = \mathcal{H}^0_{X/Z^{(2)}}(\mathcal{O}_X)$ be the $Z^{(2)}$-closure of X* [EGA$_{\mathrm{IV}}$, 5.10.16].

(i) *If X_{red} is* (S_2)*, then $X' \to X$ is finite and $X'_{\mathrm{red}} = X_{\mathrm{red}}$. In particular X'_{red} is* (S_2).
(ii) *If X is* (S_1) *and $X' \times_S S_{\mathrm{red}}$ is reduced then $X' = X$. In particular, X is* (S_2) *and $X \times_S S_{\mathrm{red}}$ is reduced.*

Proof. The question is local on S and X and we can thus assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $X' = \mathrm{Spec}(B')$. As taking reduced rings commutes with direct limits [EGA$_{\mathrm{IV}}$, Cor. 5.13.2], it follows from the definition of the $Z^{(2)}$-closure that it commutes with the reduction. In particular, if X_{red} is (S_2) then $X'_{\mathrm{red}} = X_{\mathrm{red}}$. By [EGA$_{\mathrm{IV}}$, Prop. 5.11.1], it follows that the $Z^{(2)}$-closure of X is finite if and only if the $Z^{(2)}$-closure of X_{red} is finite. As the last closure is trivial, it follows that $X' \to X$ is finite.

Now assume that X is (S_1) and $X'_{\mathrm{red}} = X' \times_S S_{\mathrm{red}}$. Then $B \to B'$ is injective and $B/\mathfrak{N}_B \to B'/\mathfrak{N}_A B'$ is an isomorphism. Thus $B' = B + \mathfrak{N}_B B'$ and it follows by Nakayama's Lemma that $B' = B$. □

Lemma (12.2). *Let S be a local artinian scheme and let X be a locally noetherian S-scheme. Let $S_1 \hookrightarrow S$ be a small nil-immersion, i.e.,* $\ker(\mathcal{O}_S \to \mathcal{O}_{S_1})\mathfrak{N}_{\mathcal{O}_S} = 0$. *Assume that X is* (S_2) *and that $X \to S$ is flat with* (S_1)*-fibers at every point $x \in X$ of codimension at most* 1*. Then $X \times_S S_1$ is* (S_1)*, i.e., has no embedded components.*

Proof. The question is local on S and X and we can thus assume that $S = \mathrm{Spec}(A)$, $S_1 = \mathrm{Spec}(A_1)$ and $X = \mathrm{Spec}(B)$. Let $I = \ker(A \to A_1)$, let $\mathfrak{N}_A$ be the nilradical of A and let $k = A/\mathfrak{N}_A$. Then $I\mathfrak{N}_A = 0$ by hypothesis and this makes I a k-module. Now let $u \in B$ such that there exists a non-zero divisor $f \in B$ with $uf \in IB$. To show that B/IB is (S_1) it is enough to show that $u \in IB$.

Let $\epsilon_1, \epsilon_2, \dots, \epsilon_n \in I$ be a k-basis of I and let $uf = \sum_i \epsilon_i b_i$ where $b_i \in B$. As f is regular and B is (S_2) we have that B/f is (S_1) [EGA$_{\mathrm{IV}}$, Cor. 5.7.6]. As $X \times_S S_{\mathrm{red}}$ has no embedded components in codimension one, it follows that the image of f in $B/\mathfrak{N}_A B$ is regular in codimension one. Thus, B/fB is flat in codimension zero as $\mathrm{Tor}_1(B/fB, A/\mathfrak{N}_A) = 0$ at points of codimension zero on $\mathrm{Spec}(B/fB)$.

Let $C = \mathrm{Tot}(B/f)$ be the total fraction ring of B/f. This is a zero-dimensional ring which is flat over A and $B/f \hookrightarrow C$. By the infinitesimal criterion of flatness, we have that the images of the b_i's in C are in $\mathfrak{N}_A C$. As $B/f \hookrightarrow C$ is faithfully flat, it follows that the images of the b_i's in B/f

are in $\mathfrak{N}_A(B/f)$, i.e., $b_i \in (f + \mathfrak{N}_A B)$. Thus $uf = \sum_i \epsilon_i f b_i'$ where $b_i' \in B$. As f is regular, it follows that $u = \sum_i \epsilon_i b_i' \in IB$. □

Proposition (12.3). *Let S be a local artinian scheme and let X be a locally noetherian S-scheme. Assume that X is* (S$_1$)*, that X_{red} is* (S$_2$)*, and that $X \to S$ is flat with reduced fibers at every point $x \in X$ of codimension at most* 1*. Then $X \times_S S_{\mathrm{red}}$ is reduced and hence* (S$_2$).

Proof. Let n be such that $\mathfrak{N}_S^n = 0$. We will show that $X \times_S S_{\mathrm{red}}$ is reduced by induction on n. If $n = 0$, then X is (R$_0$) and (S$_1$) and hence reduced.

Let $S_1 = \mathrm{Spec}(\mathcal{O}_S/\mathfrak{N}_S^{n-1})$. Let X' be the $Z^{(2)}$-closure of X. Then $X' \to X$ is an isomorphism in codimension 1. By Lemma (12.1, (i)) we have that $X'_{\mathrm{red}} = X_{\mathrm{red}}$ and $X' \to X$ is finite. In particular, X' is noetherian and (S$_2$). Thus $X' \to S$ satisfies the conditions of Lemma (12.2) and it follows that $X' \times_S S_1$ is (S$_1$). By induction, it follows that $X' \times_S S_{\mathrm{red}}$ is reduced. We now have that $X' = X$ by Lemma (12.1, (ii)) and thus that $X \times_S S_{\mathrm{red}}$ is reduced. □

Corollary (12.4). *Let S be a local artinian scheme and let X be a locally noetherian S-scheme. Assume that X is* (S$_1$) *and that $X \to S$ is flat with reduced fibers at every point $x \in X$ of codimension at most* 1*. Then $X \to S$ is flat with* (R$_0$)+(S$_2$)*-fibers at all points at which X_{red} is* (S$_2$)*. This locus is open in X.*

Proof. We can assume that $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ are affine. Let $X_{\max} = \{x_1, x_2, \dots, x_n\}$ be the generic points of X. Let $Z = \coprod_i \mathrm{Spec}(\mathcal{O}_{X,x_i}) = \mathrm{Spec}(C)$ and let $f : Z \hookrightarrow X$ be the canonical inclusion. Then f is universally schematically dominant relative to S by [EGA$_{\mathrm{IV}}$, Thm. 11.10.9] and Proposition (12.3). This means that $B \hookrightarrow C$ remains injective after tensoring with any A-algebra A'. As C is flat, we have the long exact sequence

$$0 \to \mathrm{Tor}_1^A(C/B, A') \to B \otimes_A A' \to C \otimes_A A' \to C/B \otimes_A A' \to 0$$

and it follows that C/B is flat. Thus $\mathrm{Tor}_2^A(C/B, A') = 0$ and it follows that $\mathrm{Tor}_1^A(B, A') = 0$ and hence B is flat as well. □

I recently became aware that Kollár [Kol95, Thm. 10], cf. Theorem (12.7), implies a stronger version of Corollary (12.4). When S is artinian, he shows the following. If X is (S$_1$) and $X \to S$ is flat with (S$_1$)-fibers at every point $x \in X$ of codimension at most 1, and $(X \times_S S_{\mathrm{red}})_{\mathrm{dom}/S}$ is (S$_2$), then $X \to S$ is flat with (S$_2$)-fibers. It is not difficult to modify the proofs above to obtain this result.

We now have the following generalization of a theorem of Grothendieck and Seydi [GS71, Thm. II 1]. In *loc. cit.*, only the case where S is reduced is treated.

Theorem (12.5) (Generalized Hironaka's lemma)**.** *Let S be a locally noetherian scheme. Let $f : X \to S$ be locally of finite type and* 0*-smooth. Let $x \in X$ such that $(X_{f(x)})_{\mathrm{red}}$ is geometrically normal at x. Assume that f is (locally) equidimensional and that either one of the following conditions hold:*

(i) *S is reduced and excellent.*

(ii) *f is 1-smooth.*

Then f is normal, i.e., flat with geometrically normal fibers, in a neighborhood of x.

Proof. Let $U \subseteq X$ be the open subset of f such that $f|_U$ is smooth. It is by [GS71, Prop. I 1.0] enough to show that $U \subseteq X$ is universally schematically dominant with respect to S. Moreover, it is by [GS71, Thm. I 2] enough to show this when S is the spectrum of either a local artinian ring or a discrete valuation ring, and if S is reduced and excellent only the second case is required. Note that since f is 0-flat, it is universally equidimensional.

We can thus assume that either S is the spectrum of a DVR or that S is local artinian and f is 1-smooth. The first case is the usual Hironaka lemma [$\mathrm{EGA_{IV}}$, Prop. 5.12.8]. The second case is Corollary (12.4). □

Remark **(12.6).** The excellency condition in (i) is not necessary as follows by a limit argument. Similarly, the theorem is valid without the noetherian assumption if we assume that $f : X \to S$ is locally of finite presentation.

If f is normal at x then f is 1-smooth at x. Hence the theorem shows that condition (i) implies condition (ii). Under assumption (i) of the theorem, the hypothesis that f is equidimensional is necessary as shown by Example (8.12). The hypothesis that f is equidimensional is not needed in (ii). In fact, the following theorem is a special case of Kollár's theorem [Kol95, Thm. 10].

Theorem (12.7). *Let S be a locally noetherian scheme. Let $f : X \to S$ be locally of finite type and 1-smooth. Let $x \in X$ such that $X_{f(x)}$ is (S_2) at x after removing embedded components. Then f is (S_2), i.e., flat with geometrically (S_2)-fibers, in a neighborhood of x.*

Theorem (12.8). *Let X/S be locally of finite presentation and let α be a relative cycle on X/S which is multiplicity-free. The subset of points Z_{norm} at which α is normal, is open. The morphism $Z \to S$ is normal over Z_{norm}, i.e. flat, locally of finite presentation and with geometrically normal fibers.*

Proof. Follows by Theorem (12.5). □

Corollary (12.9). *The functor $\mathrm{Hilb}_r^{\mathrm{equi}}(X) \to \mathrm{Chow}_r(X)$ induces an isomorphism between normal families of subschemes and normal families of cycles.*

Theorem (12.10). *Let X/S be locally of finite presentation and 2-smooth. Let α be a relative Weil divisor on X/S represented by the subscheme $Z \hookrightarrow X$. Let $z \in Z$ be a point over $s \in S$. If $(Z_s)_{\mathrm{emb}}$ is (S_2) at z, then $Z \to S$ is flat over z. In particular, a relative Weil divisor, parameterizing Weil divisors which are (S_2), is flat.*

Proof. Follows by Kollár's Theorem (12.7). □

Corollary (12.11). *Let X/S be flat of relative dimension $r+1$ with (R_2)-fibers. The functor $\mathrm{Hilb}_r(X) \to \mathrm{Chow}_r(X)$ induces an isomorphism between families of Cartier divisors which are (S_2) and families of Weil divisors which are (S_2).*

13. Push-forward

In this section we first define the push-forward of a (closed) relative cycle along a *finite* morphism. This definition then extends to the push-forward along a *proper* morphism, assuming that either the morphism is generically finite, i.e., that no components are collapsed under the push-forward, or that the relative cycle is represented by a flat subscheme in relative codimension one over depth zero points, e.g., the cases (A1) and (B1)–(B9) in the introduction. In particular, the proper push-forward is defined when the parameter scheme is reduced (A1) or when the relative cycle has (R_1)-fibers (B7).

Definition (13.1). Let $f : X \to Y$ be a morphism locally of finite type. We say that f is proper onto its image if $f(X)$ is locally closed and $f|_{f(X)}$ is proper.

A proper morphism is proper onto its image. A morphism which is proper at each point of $f(X)$ is proper onto its image [$\mathrm{EGA}_{\mathrm{IV}}$, Cor. 15.7.6] (at least if Y is locally noetherian).

Definition (13.2). Let $f : X \to Y$ be quasi-finite and let α be a relative cycle on X/S with support Z. Assume that $f|_Z$ is proper onto its image, e.g., that Z is closed and f is proper or that Z/S is proper and Y/S is separated. We let $f_*\alpha$ be the relative cycle on Y/S with support $f(Z)$ such that for any projection (U, B, T, p, g) of Y/S adapted to $f(Z)$ we have that $(f_*\alpha)_{U/B/T} = (\pi_1)_*\alpha_{U\times_X Y/B/T}$. Here $\pi_1 : U \times_X Y \to U$ is the projection on the first factor.

It is easily verified that $f_* \operatorname{cycl}(\alpha) = \operatorname{cycl}(f_*\alpha)$. The addition of two cycles α and β is the push-forward of $\alpha \amalg \beta$ along the morphism $X \amalg X \to X$.

(13.3) *Hilbert stack* — Let X/S be locally of finite presentation. The *Hilbert stack* $\mathscr{H}(X/S)$, parameterizes proper flat families $p : Z \to T$ equipped with a morphism $q : Z \to X$ such that $(q, p) : Z \to X \times_S T$ is quasi-finite. Even if X/S is proper, this stack is very non-separated and does not have finite automorphism groups. If X/S is separated, then the Hilbert stack is algebraic [Lie06]. It is also expected that the Hilbert stack of a non-separated scheme is algebraic [Art74, App.], in contrast to the Hilbert functor of a non-separated scheme which is not representable. Indeed, the algebraicity for zero-dimensional families is shown in [Ryd08d].

Proposition (13.4). *Let X/S be* separated *and locally of finite presentation. There is a morphism from the Hilbert stack $\mathscr{H}_r(X/S)$, parameterizing r-dimensional proper flat families, to the Chow functor* $\mathrm{Chow}_r(X/S)$. *This morphism takes a family Z/T to the relative cycle $(q, p)_*(\mathcal{N}_{Z/T})_r$ on $X \times_S T/T$.*

Remark **(13.5).** *Branchvarieties* — Let X/S be separated. The *stack of branchvarieties* of Alexeev and Knutson [AK06] is the substack of the Hilbert stack parameterizing proper and flat morphisms $p : Z \to T$ together with a morphism $q : Z \to X$ such that (q, p) is quasi-finite and p has *geometrically reduced fibers.* This stack is *proper* and has finite stabilizers but it is not Deligne-Mumford in positive characteristic. The open substack such that (q,p) is a closed immersion coincides with the open subset of the Hilbert scheme parameterizing reduced families. In particular, the morphism $\mathrm{Branch}_r(X/S) \to \mathrm{Chow}_r(X/S)$ is an isomorphism over normal embedded families, Corollary (12.9), and a monomorphism over reduced embedded families, Corollary (9.10). The morphism $\mathrm{Branch}_r(X/S) \to \mathrm{Chow}_r(X/S)$ is injective over the open locus parameterizing normal families $Z \to T$ such that $Z \to X \times_S T$ is birational onto its image. I do not know if this is a monomorphism but it seems likely.

Remark **(13.6).** *Cohen-Macaulay curves* — The space of Cohen-Macaulay curves [Høn05], is the open subset of the Hilbert stack parameterizing Cohen-Macaulay curves $Z \to T$ together with a morphism $Z \to X \times_S T$ which is birational onto its image. This is a proper algebraic space.

There is thus a plethora of moduli spaces which all maps into the Chow functor. This also includes the stack of stable maps as we will see in Corollary (13.11). To show this, we first need to define the push-forward of a relative cycle along a *proper* morphism. For simplicity we only define the push-forward for relative cycles with equidimensional support.

Definition (13.7). Let X/S and Y/S be algebraic spaces locally of finite type over S. Let $f : X \to Y$ be a morphism and let $Z \subseteq X$ be a locally closed subset such that Z/S is equidimensional and $f|_Z$ is proper onto its image. Let $U \subseteq f(Z)$ be the open subset over which $Z \to f(Z)$ is quasi-finite. We let $f_*(Z) \subseteq f(Z)$ be the closure of U.

Remark **(13.8).** If $f|_Z : Z \to Y$ is quasi-finite at the generic points of Z, then $f_*(Z) = f(Z)$.

Lemma (13.9). *Let $f : X \to Y$ be a morphism and let $Z \subseteq X$ be a locally closed subset such that Z/S is equidimensional and such that $f|_Z$ is proper onto its image. Let $U \subseteq f(Z)$ be the open subset over which $Z \to f(Z)$ is quasi-finite. Then $f_*(Z)$ is equidimensional over S and $U \subseteq f_*(Z)$ is fiberwise dense. In particular, $f_*(Z)$ commutes with base change, i.e., for any morphism $g : S' \to S$ we have that*

$$g^{-1}(f_*(Z)) = f'_*(g^{-1}(Z))$$

Proof. Let $s \in S$ and let $y \in f_*(Z)_s$ be a generic point and let r be the dimension of $f_*(Z)_s$ at y. Let $W \subseteq f_*(Z)$ be an irreducible component containing y and let $V \subseteq Z$ be an irreducible component mapping onto W such that $V \to W$ is quasi-finite. Then as V is equidimensional, it follows that W is equidimensional at y and that $V \to W$ is quasi-finite over y. This shows that $y \in U$. □

Theorem (13.10). *Let S be locally noetherian, let X/S and Y/S be locally of finite type, let $f : X \to Y$ be a morphism and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module which is flat over S. Let $Z = \mathrm{Supp}(\mathcal{F})$ and assume that $f|_Z$ is proper onto its image and that Z/S is equidimensional. Let $U \subseteq f(Z)$ be the open subset over which $Z \to f(Z)$ is quasi-finite. Let $V = f^{-1}(U)$. Then the* *relative cycle* $f_*\big(\mathcal{N}_{\mathcal{F}|_V}\big)$ *on* U/S *extends uniquely to a relative cycle on* Y/S *with support* $f_*(Z)$*. This cycle is denoted by* $f_*\mathcal{N}_{\mathcal{F}/S}$*.*

Proof. Replacing X with the closed subscheme defined by $\operatorname{Ann}_{\mathcal{O}_X}(\mathcal{F})$, and Y with its image, we can assume that f is proper.

First assume that f is not only proper but also projective. Let $\mathcal{L}$ be an invertible sheaf on X which is f-ample. Then for sufficiently large n, we have that $\mathrm{R}^i f_*(\mathcal{F}\otimes\mathcal{L}^n)=0$ for all $i>0$ and that $\mathcal{G}=f_*(\mathcal{F}\otimes\mathcal{L}^n)$ is coherent and flat over S. As $f_*(\mathcal{F}\otimes\mathcal{L}^n)|_U$ and $f_*(\mathcal{F})|_U$ are locally isomorphic, it follows that $\mathcal{N}_{\mathcal{G}}$ is an extension of $f_*(\mathcal{N}_{\mathcal{F}|_V})$.

In general, $\mathbf{R}f_*(\mathcal{F})$ is a perfect complex relative to S [SGA$_6$, Exp. III, Prop. 4.8] and $\mathcal{N}_{\mathbf{R}f_*(\mathcal{F})/S}$ is the required extension, cf. Remark (7.11). □

Corollary (13.11). *Let* X/S *be separated and locally of finite presentation. For any genus* g*, number of marked points* n *and homology class* β*, there is a functor from the Kontsevich space* $\overline{\mathscr{M}}_{g,n}(X/S,\beta)$ *of stable maps into* X *to the Chow functor* $\mathrm{Chow}_1(X/S)$ *taking a stable curve onto its image cycle.*

Theorem (13.12). *Let* $f:X\to Y$ *be a morphism and let* α *be a relative cycle on* X/S *with equidimensional support* Z *such that* $f|_Z$ *is proper onto its image. Let* $U\subseteq f(Z)$ *be the open subset over which* $Z\to f(Z)$ *is quasi-finite. Then there is at most one extension of* $f_*(\alpha|_{f^{-1}(U)})$ *to* $f_*(Z)$*. When such an extension exists, we denote it by* $f_*\alpha$*. An extension exists if one of the following conditions is satisfied:*

(1) $Z\to f(Z)$ *is quasi-finite at points* $y\in f_*(Z)$ *such that* y *has codimension one in a fiber over a point of depth zero in* S*.*

(1a) $f|_Z$ *is generically finite, i.e.,* $f_*(Z)=f(Z)$*.*

(2) *There is an open subset* $V\subseteq Z$ *containing all points* $x\in Z$ *of relative codimension one over points of depth zero of* S*, such that* $\alpha|_V=\mathcal{N}_{V_1/S}$ *where* $V_1\to S$ *is flat and finitely presented.*

(2a) S *is reduced.*

(2b) α *has* (R_1)*-fibers.*

(2c) $X\to S$ *is* 2*-smooth, e.g.* (R_2)*, and* α *is a relative Weil divisor.*

Proof. Note that (1a) is a special case of (1) and that (2a)–(2c) are special cases of (2). By Lemma (13.9), the open subset $U\subseteq f(Z)$ contains all points of $f_*(Z)$ which are generic in their fibers over S. By Corollary (6.6) there is thus at most one extension and an extension to $f_*(Z)$ exists if an extension to all points of $f_*(Z)$ which are of codimension one in its fiber over a point $s\in S$ of depth zero exist. In (1) all such points are already in U and in (2) an extension exists by Theorem (13.10). □

Conditions (2a)–(2c) contain the cases (A1) and (B1)–(B9) of the introduction. It is likely that $f_*\alpha$ always is defined, i.e., that $f_*(\alpha|_{f^{-1}U})$ always extends to $f_*(Z)$.

14. Flat pull-back and products of cycles

Let S be a locally noetherian scheme, let X/S be locally of finite type and let α be a relative cycle on X/S with support Z. Let $f:Y\to X$ be a flat morphism, locally of finite presentation. We would like to define the pull-back $f^*\alpha$ of α as a relative cycle on Y/S. The pull-back should satisfy the following two conditions

(P1) $f^* \operatorname{cycl}(\alpha) = \operatorname{cycl}(f^*\alpha)$.
(P2) If $\mathcal{F}$ is a coherent sheaf on X which is flat over S, then $f^*\mathcal{N}_{\mathcal{F}/S} = \mathcal{N}_{f^*\mathcal{F}/S}$.

Note that $f^* \operatorname{cycl}(\mathcal{N}_{\mathcal{F}/S}) = \operatorname{cycl}(\mathcal{N}_{f^*\mathcal{F}/S})$ so these two conditions are compatible. When one of the following conditions holds

(A1) S is reduced.
(A2) α is multiplicity-free.
(A3) α is a relative Weil divisor and X/S is 1-smooth.

then there is at most one relative cycle $f^* \operatorname{cycl}(\alpha)$ satisfying (P1)–(P2) by Corollary (6.6) and the results of Sections 8–9. Similarly, we obtain the following result from Corollary (6.6).

Proposition (14.1). *Let S be arbitrary and let α be a relative cycle on X/S with support Z. Let $f : Y \to X$ be a flat morphism. Assume that there exists an open subset $U \subseteq Z$ containing all points $z \in Z$ over $s \in S$ with* $\operatorname{codim}_z Z_s + \operatorname{depth}_s S \leq 1$*, such that α is represented by a flat subscheme or cycle over U, cf.* (B1), (B3)–(B9) *in the introduction. Then there is a unique cycle $f^* \operatorname{cycl}(\alpha)$ satisfying* (P1)–(P2).

The proposition is also valid when S is semi-normal and α is arbitrary, i.e., in the case (B2). This follows from Corollary (10.19) and the following discussion.

Let us now discuss the general case. Locally on Y there exists a factorization $Y \to U \to X$ of f such that the first morphism is quasi-finite and the second morphism is smooth. By Lemma (7.12), $Y \to U$ is of finite Tor-dimension. Locally, $U \to X$ factors through an étale morphism $U \to \mathbb{A}^n_X$. If $(U, B, \mathbb{A}^n_S)$ is a projection adapted to Z/S we then define

$$(f^*\alpha)_{Y/B/S} = \alpha_{U/B/\mathbb{A}^n_S} \circ \mathcal{N}_{Y/U}.$$

If α satisfies condition $(*)$ of Section 5, then $(f^*\alpha)_{Y/B/S}$ does not depend upon the morphism $B \to \mathbb{A}^n_S$ and is thus well-defined. Now, the problem is that a general smooth projection (Y, B, S) adapted to $f^{-1}(Z)$ does not admit such a factorization.

If S is of *characteristic zero* and f is *smooth*, then any smooth projection (Y, B, S) generically admits such a factorization. Indeed, let n be the relative dimension of $Y \to X$ and let $B \to \mathbb{A}^{r+n}_S$ be an étale morphism. Then the induced morphism $Y \to X \times_S \mathbb{A}^{r+n}_S$ is quasi-finite. Thus, there is a projection $\mathbb{A}^{r+n}_S \to \mathbb{A}^n_S$ such that $Y \to X \times_S \mathbb{A}^n_S$ is quasi-finite and hence generically étale fiberwise over X.

Now assume as before that f is smooth but let S be arbitrary. Then there exists smooth projections (Y, B, S) such that $Y \to X \times_S \mathbb{A}^n_S$ cannot be chosen so that it is generically étale. For example, let k be a field of characteristic p, $X = S = \operatorname{Spec}(k)$, $Y = \operatorname{Spec}(k[t])$ and $B = \operatorname{Spec}(k[t^p])$. For a *generic* choice of (Y, B, S) we can however find a factorization $Y \to X \times_S \mathbb{A}^n_S$ which is generically étale fiberwise over X.

Theorem (14.2). *Let S be an arbitrary scheme, let α be a relative cycle on X/S with support Z and let $f : Y \to X$ be a* smooth *morphism. Assume that α is represented by a flat subscheme or a flat cycle over an open subset $U \subseteq Z$ containing all points of relative codimension at most one over points of depth zero in S. This is the case if S is reduced or if α is as in* (B1)–(B9) *of the introduction. Then there is a unique relative cycle $f^*\alpha$ on Y/S satisfying* (P1)–(P2)*. Furthermore, for every commutative diagram*

$$\begin{array}{ccccc} U & \xrightarrow{p} & Y & \xrightarrow{f} & X \\ \downarrow & & & & \downarrow \\ B & & & & \\ \downarrow & & & & \\ T' & \xrightarrow{g'} & T & \xrightarrow{g} & S \end{array}$$

with p, g, g' smooth, $U \to Y \times_S T'$ étale and $U \to X \times_S T$ étale, we have that

$$(f^*\alpha)_{U/B/T'} = \alpha_{U/B/T}.$$

Proof. First note that since α is represented by a flat subscheme or flat cycle in relative codimension zero over depth zero points, α satisfies condition $(*)$. Let (U, B, T', p, g'') be a smooth projection of Y/S. As discussed above, there is then a factorization of $g'' : T' \to S$ into smooth morphisms $g' : T' \to T$ and $g : T \to S$ such that $U \to X \times_S T$ is quasi-finite (but not necessarily generically étale in characteristic p). Picking a generic smooth projection and placing the two projections in a family, we obtain a smooth projection and morphisms as above such that $U \to X \times_S T$ is generically étale, fiberwise over X, and such that the original projection is obtained as a pull-back of this family.

As α is represented by a flat subscheme or flat cycle in relative codimension at most one over depth zero points, it follows that the common definition of $f^*\alpha$ at points of relative codimension zero over depth zero points extends. □

Similarly, we would like to define *products* of cycles, i.e., if α is a relative cycle on X/S and β is a relative cycle on Y/S we would like to define $\alpha \times \beta$ on $X \times_S Y/S$. This relative cycle should satisfy obvious conditions such as $\mathrm{cycl}(\alpha \times \beta) = \mathrm{cycl}(\alpha) \times \mathrm{cycl}(\beta)$ and $\mathrm{cycl}(\mathcal{N}_{\mathcal{F}} \times \mathcal{N}_{\mathcal{G}}) = \mathrm{cycl}(\mathcal{N}_{\mathcal{F}\otimes\mathcal{G}})$. When α and β are as in (A1)–(A3) then there is at most one such product cycle and when α and β are as (B1)–(B9) there exists a product cycle, cf. Proposition (14.1). I do not know if it is possible to employ similar methods as in Theorem (14.2) to show the existence of a product cycle when S is reduced.

15. Projections and intersections

Proposition (15.1) (Projection)**.** *Let α be a relative cycle on X/S, let $Y \to S$ be smooth and let $X \to Y$ be a morphism such that $\mathrm{Supp}(\alpha)|_{\mathrm{dom}/Y} \to Y$ satisfies* (T)*. Then there is an induced relative cycle α' on X/Y such that* *for any projection* $(U, B, \mathbb{A}^r_Y)$ *of* X/Y *adapted to* $\operatorname{Supp}(\alpha')$ *we have that* $\alpha'_{U/B/\mathbb{A}^r_Y} = \alpha_{U/B/\mathbb{A}^r_S}$.

Proof. This follows from the fact, Proposition (5.3), that it is enough to consider projections of the form $(U, B, \mathbb{A}^r_Y)$ to define α'. $\square$

Definition (15.2). Let α be a relative cycle on X/S with support Z. Let $\mathcal{L}$ be a invertible sheaf on X/S and let $f \in \Gamma(X, \mathcal{L})$ be a global section. Assume that the closed subscheme $V(f)$ defined by f intersects Z properly in every fiber, i.e., that $V(f)_s$ does not contain an irreducible component of Z_s for every $s \in S$. Locally on X, the section f induces a projection $(X, \mathbb{A}^1, S)$ and hence a relative cycle α' on $X/\mathbb{A}^1$. We let $\mathcal{L}_f \cap \alpha$ be the relative cycle with support $Z \cap V(f)$ defined locally on X as the pull-back of α' along the zero-section of $\mathbb{A}^1 \to S$.

In particular, if D is a relative Cartier divisor on X/S which intersects Z properly, then we let $D \cap \alpha = \mathcal{O}(D)_f \cap \alpha$ where f is the section given as the dual of $\mathcal{I}_D = \mathcal{O}(-D) \hookrightarrow \mathcal{O}_X$.

If $f_1, f_2, f_3, \dots, f_n$ is a sequence of sections of $\mathcal{O}_X$ such that $V(f_i)$ intersects $V(f_{i-1}) \cap \dots \cap V(f_1) \cap Z$ properly for $i = 1, 2, \dots, n$, then $V(f_n) \cap V(f_{n-1}) \cap \dots \cap V(f_1) \cap \alpha$ is defined. It is clear that this relative cycle, which we denote by $V(f_1, f_2, \dots, f_n) \cap \alpha$, does not depend upon the ordering of the f_i's. On the other hand, if $g_1, g_2, \dots, g_n$ is another sequence such that the relative cycle $V(g_1, g_2, \dots, g_n) \cap \alpha$ exists and $(f_1, f_2, \dots, f_n) = (g_1, g_2, \dots, g_n)$ as ideals, then it is not clear that the corresponding cycles coincide.

Assume that $V(f_1, f_2, \dots, f_n) \cap \alpha$ only depends on the ideal $(f_1, f_2, \dots, f_n)$ in general. If $Y \hookrightarrow X$ is a regular immersion intersecting $Z = \operatorname{Supp}(\alpha)$ properly, we can then define a relative cycle $Y \cap \alpha$ locally using any regular sequence defining Y. Under this assumption, we can now define proper intersections of relative cycles on smooth schemes:

Definition (15.3). Let X/S be smooth of relative dimension n and let α, β be relative cycles on X/S, equidimensional of dimensions r and s respectively. Assume that $\operatorname{Supp}(\alpha)$ and $\operatorname{Supp}(\beta)$ intersect properly in each fiber, i.e., that $\operatorname{Supp}(\alpha) \cap \operatorname{Supp}(\beta)$ is equidimensional of dimension $r+s-n$. Then $\operatorname{Supp}(\alpha) \cap \operatorname{Supp}(\beta) = \Delta_{X/S} \cap \big(\operatorname{Supp}(\alpha) \times \operatorname{Supp}(\beta)\big)$ and the latter intersection is proper in each fiber. We let $\alpha \cap \beta = \Delta_X \cap (\alpha \times \beta)$ when the relative cycle $(\alpha \times \beta)$ is defined, cf. Section 14.

16. Relative fundamental classes of relative cycles

We briefly indicate the construction of relative fundamental classes and the relation with Angéniol's functor. Throughout this section, S is a locally noetherian scheme over $\operatorname{Spec}(\mathbb{Q})$ and X/S is of finite type and *separated*.

Theorem (16.1). *Let α be a relative cycle on X/S which is equidimensional of dimension r. Then there exists an infinitesimal neighborhood $j : Z \hookrightarrow X$ of* $\operatorname{Image}(\alpha) \hookrightarrow X$ *such that α is the push-forward of a relative cycle on Z along j. Moreover, there is a class $c_\alpha \in \operatorname{Ext}^{-r}_Z\big(\Omega^r_{Z/S}, \mathcal{D}^\bullet_{Z/S}\big)$, the* relative fundamental class *of α, such that for any projection (U, B, T, p, g) the composition of the canonical homomorphism $h_* h^* \Omega^r_{B/S} \to h_* \Omega^r_{Z/S}$ and* *the trace* $\operatorname{tr}(c_\alpha) : h_*\Omega^r_{Z/S} \to \Omega^r_{B/S}$, *coincides with the trace* $h_*\mathcal{O}_{p^{-1}(Z)} \to \mathcal{O}_B$ *induced by* $\alpha_{U/B/T}$ *after tensoring with* $\Omega^r_{B/S}$. *Here* h *denotes the morphism* $p^{-1}(Z) \to U \to B$.

Proof. This can be proved using Bott's theorem on grassmannians almost exactly as in [AEZ78, §III]. We indicate the steps.

Note that Z is not unique, but if we have obtained Z and c_α on an open cover, then we can take a common infinitesimal neighborhood of the Z's and on this neighborhood the c_α's glue. We can thus assume that X and S are affine.

Let $Z_0 = \operatorname{Supp}(\alpha)$ and take an embedding $X \hookrightarrow \mathbb{A}^n$ as in Lemma (9.6) and consider the corresponding universal projection $(\mathbb{A}^n_{\mathbb{G}}, \mathbb{B}, \mathbb{G}, p)$. To simplify the presentation, we will now let $X = \mathbb{A}^n$. Let $U \subseteq Z_0 \times_S \mathbb{G}$ be the open subset over which $Z_0 \times_S \mathbb{G} \hookrightarrow \mathbb{A}^n_{\mathbb{G}} \to \mathbb{B}$ is quasi-finite. The subset U then contains all points of relative codimension at most one over Z_0 by Lemma (9.6).

Let $Z \hookrightarrow \mathbb{A}^n_S$ be the image of $\operatorname{Image}(\alpha_{U/\mathbb{B}/\mathbb{G}})$ along p, and denote the inclusion with i. Let $h : Z \times_S \mathbb{G} \to \mathbb{B}$ be the corresponding morphism, let $Z' = U \cap Z \times_S \mathbb{G}$ and denote the open immersion $Z' \hookrightarrow Z \times_S \mathbb{G}$ with j. On Z we have the sheaf $i^*(\Omega^1_{\mathbb{A}^n_S})^\vee$ which is free of rank n. Thus, we have that $Z \times_S \mathbb{G} = \mathbb{G}_r(i^*(\Omega^1_{\mathbb{A}^n_S})^\vee)$. Let $\mathcal{H}$ be the universal quotient sheaf on $Z \times_S \mathbb{G}$. It is then readily verified that there is a natural isomorphism $\mathcal{H} \cong h^*(\Omega^1_{\mathbb{B}/\mathbb{G}})^\vee$ [AEZ78, §I.3].

Let $\mathcal{W} = \mathcal{E}xt^{-r}(\mathcal{O}_Z, \mathcal{D}^\bullet_{Z/S})$. The relative zero-cycle $\alpha_{U/\mathbb{B}/\mathbb{G}}$ induces a global section of

$$\begin{aligned}\mathcal{E}xt^{-r}_{Z'}\big(j^*h^*(\Omega^r_{\mathbb{B}/\mathbb{G}}), \mathcal{D}^\bullet_{Z'/\mathbb{G}}\big) &= \mathcal{E}xt^{-r}_{Z'}\big(j^*(\wedge^r\mathcal{H}^\vee), p^*\mathcal{D}^\bullet_{Z/S}\big)\\ &= j^*(\wedge^r\mathcal{H}) \otimes_{\mathcal{O}_{Z'}} p^*\mathcal{W}\end{aligned}$$

by (2.27). Bott's theorem [AEZ78, Cor. I.2] shows that the canonical homomorphism

$$\wedge^r i^*(\Omega^1_{\mathbb{A}^n_S})^\vee \otimes \mathcal{W} \to p_*\big(j^*(\wedge^r\mathcal{H}) \otimes p^*\mathcal{W}\big)$$

is an isomorphism. We thus obtain a global section of

$$i^*(\Omega^r_{\mathbb{A}^n_S})^\vee \otimes \mathcal{W} = \mathcal{E}xt^{-r}\big(i^*\Omega^r_{\mathbb{A}^n_S}, \mathcal{D}^\bullet_{Z/S}\big)$$

and this is the relative fundamental class of α as it can be shown that this factors through $\Omega^r_{Z/S}$. What remains is to show that for any projection (U, B, S) the trace $c_{U/B/S}$ is the trace of the zero-cycle $\alpha_{U/B/S}$. This is done almost exactly as in [AEZ78, §III]. □

Let X/S be smooth of relative dimension n and let $Z \hookrightarrow X$ be a closed subset which is equidimensional of dimension r over S. Let c be a class in $\mathrm{H}^{n-r}_Z(X, \Omega^{n-r}_{X/S})$. This class lifts to a class in $\operatorname{Ext}^{-r}_X((j_m)_*\mathcal{O}_{Z_m}, \Omega^{n-r}_{X/S})$ for some infinitesimal neighborhood $j_m : Z_m \hookrightarrow X$ of Z.

If (U, B, T, p, g, φ) is any smooth projection adapted to Z, there is an induced trace homomorphism $\operatorname{tr}(c) : \varphi_*(j_m)_*\Omega^r_{p^{-1}(Z_m)/T} \to \Omega^r_{B/T}$ which induces a homomorphism $\operatorname{tr}(c) : \varphi_*(j_m)_*\mathcal{O}_{Z_m} \to \mathcal{O}_B$.

Now if c is a *Chow class* [Ang80, Def. 4.1.2] then $\mathrm{tr}(c)$ is the trace corresponding to a relative zero-cycle $c_{U/B/T}$ on U with image contained in Z_m by [Ang80, Prop. 2.3.5 and Thm. 1.5.3]. These zero-cycles define a relative cycle on X/S.

Theorem (16.2). *The morphism from Angéniol's Chow-space* $\mathrm{Ang}_r(X/S)(T)$ *to the Chow functor* $\mathrm{Chow}_r(X/S)(T)$ *taking a Chow class onto the corresponding relative cycle is a* monomorphism. *When T is reduced, or when restricted to multiplicity-free cycles or relative Weil-divisors, this morphism is an isomorphism.*

Proof. Let c be a Chow class. Then c is determined by the induced relative zero-cycles $c_{U/B/T}$. In fact, c is determined by $c_{U/\mathbb{B}/\mathbb{G}}$ for a universal projection as in the proof of Theorem (16.1).

Let α be a relative cycle on X/S and let (U, B, S, p, g, φ) be a smooth projection. The corresponding class c is a Chow class if the trace homomorphism $\varphi_*(j_m)_*\Omega^r_{Z_m/S} \to \Omega^r_{B/S}$ satisfies the conditions of [Ang80, Thm. 4.1.1]. These conditions can be checked on depth zero points of B.

In the three special cases listed, $\alpha_{U/B/S}$ is represented by a flat subscheme or a flat cycle on a schematically dense open subset of B. That c is a Chow class then follows from [Ang80, Prop. 7.1.1]. $\square$

Corollary (16.3). *Let X be a quasi-projective scheme over $\mathbb{C}$. Then the reduction of the Chow functor* $\mathrm{Chow}_r(X)$ *is represented by the Chow variety* $\mathrm{ChowVar}_{r,d}(X)$.

Proof. By the theorem, we have that $\mathrm{Chow}_r(X)_{\mathrm{red}} = \mathrm{Ang}_r(X)_{\mathrm{red}}$ and the latter space coincides with the Barlet space [Ang80, Thm. 6.1.1]. As the Barlet space coincides with the Chow variety [Bar75, Ch. IV, Cor. of Thm. 7] the result follows. $\square$

17. The classical Chow embedding and representability

In this section, we briefly review the classical construction of the Chow variety, cf. [CW37, Sam55, GKZ94, Kol96], and the extension of this construction to arbitrary relative cycles.

(17.1) *The incidence correspondence* — Let S be a scheme and $\mathcal{E}$ a locally free sheaf on $\mathcal{O}_S$ of rank $N+1$. There is a natural commutative square

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \xleftarrow{p} & \mathbb{F}_{1,N-r}(\mathcal{E}) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle q} \\ S & \longleftarrow & \mathbb{G}_{N-r}(\mathcal{E}) \end{array}$$

where $G = \mathbb{G}_{N-r}(\mathcal{E})$ is the grassmannian parameterizing linear subvarieties of codimension $r+1$ in $\mathbb{P}(\mathcal{E})$ and $I = \mathbb{F}_{1,N-r}(\mathcal{E})$ is the flag variety parameterizing linear subvarieties of codimension $r+1$ with a marked point. The morphisms p and q are grassmannian fibrations and in particular smooth.

If $Z \hookrightarrow \mathbb{P}(\mathcal{E})$ is equidimensional of dimension r, then $p^{-1}(Z)$ is equidimensional of dimension $r + (N-r-1)(r+1) = (N-r)(r+1) - 1$. It is easily seen that $q|_{p^{-1}(Z)}$ is generically finite, fiberwise over S, and thus

$\mathrm{CH}(Z) := q(p^{-1}(Z))$ is a hypersurface in $\mathbb{G}$. If $S = \mathrm{Spec}(k)$, and Z has degree d, then $\mathrm{CH}(Z)$ has degree d with respect to the Plücker embedding of G. Note that if $S = \mathrm{Spec}(k)$, then $\mathrm{CH}(Z)$ is a Cartier divisor on $\mathbb{G}$.

Remark **(17.2).** A variant of the incidence correspondence is often used, cf. [CW37, GIT, Kol96]. Instead of grassmannians and flag varieties, we take G as the multi-projective space $\mathbb{P}(\mathcal{E}^\vee)^{r+1}$ and I as the subscheme of $\mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}^\vee)^{r+1}$ given as the intersection of the $r+1$ universal hyperplanes. Then $q(p^{-1}(Z))$ becomes a hypersurface of multi-degree $d, d, \dots, d$ in G.

(17.3) *The Chow variety* — Using Chevalley's theorem on the semi-continuity of the fiber dimension, it is easily seen that there is a closed subset of $\mathrm{Div}^d(G/S)$ corresponding to cycles on $\mathbb{P}(\mathcal{E})$ of dimension r and degree d [Kol96, I.3.25.1].

Definition (17.4). Let $X \hookrightarrow \mathbb{P}(\mathcal{E})$ be a subscheme and let α be a proper equidimensional relative cycle of dimension r on X/S such that the smooth pull-back is defined, cf. (14.2), then we let $\mathrm{CH}(\alpha) = q_* p^*(\alpha)$.

Note that q is generically finite over $p^{-1}(\mathrm{Supp}(\alpha))$ so the existence of q_* follows by Theorem (13.12). As $\mathrm{CH}(\alpha)$ is a relative Weil divisor and G/S is smooth, we obtain by Theorem (9.15) a morphism

$$\mathrm{CH} : \mathrm{Chow}_{r,d}(X/S) \to \mathrm{Div}^d(G/S).$$

If α is a relative cycle on X/S, then $\mathrm{cycl}(\mathrm{CH}(\alpha)) = q_* p^*(\mathrm{cycl}(\alpha))$ so this morphism extends the usual map of cycles. If $Z \hookrightarrow X$ is a closed subscheme which is flat and proper over S, then

$$\mathrm{CH}\left(\mathcal{N}_{Z/S}\right) = \mathrm{CH}\left(\mathcal{N}_{q_*(p^{-1}(Z) \otimes \mathcal{L})}\right)$$

for some sufficiently q-ample sheaf $\mathcal{L}$ on I. That the corresponding Cartier divisor coincides with the divisor constructed by Mumford [GIT, Ch. 5, §3] with the Div-construction follows from Proposition (7.13).

Proposition (17.5). *Let X/S be a quasi-projective scheme with a projective embedding morphism $X \hookrightarrow \mathbb{P}(\mathcal{E})$. Let α be a relative cycle on X/S, equidimensional of dimension r. Assume that one of the following holds:*

(i) *S is reduced.*
(ii) *X/S is of relative dimension $r+1$ and 1-smooth.*
(iii) *α is multiplicity-free.*

Then α can be recovered from $\mathrm{CH}(\alpha)$.

Proof. If S is reduced, it is enough to show that α_s can be recovered for any generic point s. We can thus assume that S is the spectrum of a algebraically closed field. As the CH-morphism is additive, we can also assume that α corresponds to an irreducible variety V. Then $V = X \setminus p(q^{-1}(G \setminus \mathrm{CH}(V)))$.

Under the hypothesis in (ii) and (iii), α is represented by a subscheme $Z \hookrightarrow X$ which is either a relative Cartier divisor or smooth over S on a schematically dense subset U of Z. To show that α can be recovered from $\mathrm{CH}(\alpha)$ it is enough to construct $Z|_U$. This can be done as in [Fog71, §5]. □

Questions **(17.6).** In characteristic zero, we have that

$$\mathrm{Chow}_{r,d}(X/S)_{\mathrm{red}} = \mathrm{Ang}_{r,d}(X/S)_{\mathrm{red}} = \mathrm{ChowVar}_{r,d}(X/S)$$

and thus the morphism $\mathrm{Chow}_{r,d}(X/S)_{\mathrm{red}} \to \mathrm{Div}^d(G)$ is an immersion. This leads to the following questions:

- Is $\mathrm{Ang}_{r,d}(X/S) \to \mathrm{Div}^d(G)$ an immersion?
- In positive characteristic, is $\mathrm{Chow}_{r,d}(X/S) \to \mathrm{Div}^d(G)$ an immersion for sufficiently ample embeddings $X \hookrightarrow \mathbb{P}(\mathcal{E})$?
- In positive characteristic, is $\mathrm{Chow}_{r,d}(X/S)_{\mathrm{red}} \to \mathrm{Div}^d(G)$ an immersion for sufficiently ample embeddings $X \hookrightarrow \mathbb{P}(\mathcal{E})$?

Appendix A. Duality and fundamental classes

Let $f : X \to S$ be a morphism of schemes. We assume that S is noetherian and that f is separated and of finite type. Then f admits a compactification, i.e. there is a proper morphism $\overline{X} \to S$ and a schematically dominant immersion $X \hookrightarrow \overline{X}$ of S-schemes. This is a famous theorem by Nagata [Nag62, Lüt93]. Nagata's compactification result has been generalized by Raoult to algebraic spaces when either X is normal or S is the spectrum of a field [Rao71, Rao74] but we do not need this.

Using that separated, finite type morphisms are compactifiable, one constructs a pseudo-functor !, the twisted (or extraordinary) inverse image, from the category of noetherian schemes and finite type separated morphisms to the corresponding derived category. If $f : X \to S$ is a finite type separated morphism of noetherian schemes then $f^!(\mathcal{O}_S) = \mathcal{D}^\bullet_{X/S}$ is the *relative dualizing complex* constructed by Deligne [Har66, App. by Deligne]. If $g : U \to X$ is an étale morphism then $g^! = g^*$ and if $f : X \to S$ is a *proper* morphism, then $f^!$ is a right adjoint to f_* (in the derived category). If f is a finite type separated morphism of finite Tor-dimension, then $f^!(\mathcal{F}) = f^*(\mathcal{F}) \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathcal{D}^\bullet_{X/S}$. If $f : X \to S$ is smooth of relative dimension r, then $\mathcal{D}^\bullet_{X/S} = \Omega^r_{X/S}[r]$.

As $g^! = g^*$ for étale morphisms, we can extend the definition of ! to the category of noetherian algebraic spaces with finite type separated morphisms. We will use the following duality theorem:

Theorem (A.1) ([Har66, Ch. III, Thm. 6.7]). *Let $f : X \to Y$ be a finite morphism of noetherian schemes. Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be complexes of sheaves on X and Y respectively. Then there is a quasi-isomorphism*

$$f_*\mathbf{R}\mathcal{E}xt_X(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet) \to \mathbf{R}\mathcal{E}xt_Y(f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

In particular, we have that

$$\mathrm{Ext}^n_X(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet) \to \mathrm{Ext}^n_Y(f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$

for every integer n.

We briefly recall some of the main results of [EZ78]. Let k be a field. If X/k is smooth of dimension r, then the *fundamental class* of X/k is the canonical class

$$c_X \in \mathrm{Ext}^{-r}_X(\Omega^r_{X/k}, \mathcal{D}^\bullet_{X/k}) = \mathrm{Hom}_X(\Omega^r_{X/k}, \Omega^r_{X/k})$$

given by the identity. If X/k is geometrically reduced, then there is a unique class

$$c_X \in \operatorname{Ext}_X^{-r}(\Omega^r_{X/k}, \mathcal{D}^\bullet_{X/k}),$$

the fundamental class of X/k, such that over the smooth locus $U \subseteq X$, the pull-back $c_X|_U = c_U$ coincides with the class defined above. The uniqueness of c_X follows by Corollary (A.4) below. The existence of the fundamental class c_X for arbitrary X/k is shown by El Zein [EZ78, Ch. III, Thm.]. When X/k is not geometrically reduced, c_X is uniquely determined as follows. If the irreducible components of X are X_i, then

$$c_X = \sum_i m_i c_{X_i}$$

where m_i is the multiplicity of X_i, i.e., the length of the local ring of $\mathcal{O}_X$ at the generic point of X_i, cf. Remark (A.8). If K/k is a perfect extension of k, then

$$\operatorname{Ext}_X^{-r}(\Omega^r_{X/k}, \mathcal{D}^\bullet_{X/k}) \to \operatorname{Ext}_{X_K/K}^{-r}(\Omega^r_{X_K/K}, \mathcal{D}^\bullet_{X_K/K})$$

is injective and the image of c_X is c_{X_K} [EZ78, Ch. III, No. 4, Prop.]. Note that when k is of characteristic $p > 0$, then c_X is zero at every irreducible component X_i where p divides the geometric multiplicity, i.e. the multiplicity of $(X_i)_K$.

Assume that X can be smoothly embeddable, i.e., that there exists a closed immersion $j : X \hookrightarrow Y$ into a smooth scheme Y/k of pure dimension n. Then we define the *algebraic de Rham homology* of X by

$$\mathrm{H}_q^{\mathrm{dR}}(X) = \mathbf{H}_X^{2n-q}(Y, \Omega^\bullet_{Y/k}),$$

the hypercohomology, with supports in X, of the algebraic de Rham complex on Y. If k has characteristic zero, then this homology group is independent on the choice of smooth embedding [Har75, Ch. II, Thm. 3.2].

We have a canonical homomorphism

$$\begin{aligned}
\operatorname{Ext}_X^{-r}(\Omega^r_{X/k}, \mathcal{D}^\bullet_{X/k}) &\to \operatorname{Ext}_X^{-r}(j^*\Omega^r_{Y/k}, j^!\Omega^n_{Y/k}[n]) \\
&\cong \operatorname{Ext}_Y^{-r}(j_*j^*\Omega^r_{Y/k}, \Omega^n_{Y/k}[n]) \\
&\cong \operatorname{Ext}_Y^{n-r}(j_*\mathcal{O}_X, \Omega^{n-r}_{Y/k}) \\
&\to \mathrm{H}_X^{n-r}(Y, \Omega^{n-r}_{Y/k}).
\end{aligned}$$

By abuse of notation, we also denote the image of the fundamental class c_X in $\mathrm{H}_X^{n-r}(Y, \Omega^{n-r}_{Y/k})$ by c_X. El Zein [EZ78, Ch. III, Thm.] shows that c_X is in the kernel of the differential

$$d' : \mathrm{H}_X^{n-r}(Y, \Omega^{n-r}_{Y/k}) \to \mathrm{H}_X^{n-r}(Y, \Omega^{n-r+1}_{Y/k}).$$

Thus c_X is the image of an element in the hypercohomology

$$\mathbf{H}_X^{n-r}(Y, \Omega^{n-r}_{Y/k} \to \Omega^{n-r+1}_{Y/k} \to \dots)$$

which we also denote by c_X. Finally, we have the image of this element in the algebraic de Rham homology:

$$\mathbf{H}_X^{2(n-r)}(Y, \Omega^\bullet_{Y/k}) = \mathrm{H}_{2r}^{\mathrm{dR}}(X).$$

In characteristic zero, this class coincides with the homology class $\eta(X)$ defined by Hartshorne [Har75, Ch. II, 7.6]. This is not proved by El Zein but not difficult to show. In fact, as $c_X = \sum_i m_i c_{X_i}$ where X_i are the irreducible components of X and m_i their multiplicities, we can assume that X is integral. Then $\mathrm{H}^{\mathrm{dR}}_{2r}(X) \cong \mathrm{H}^{\mathrm{dR}}_{2r}(X \setminus X_{\mathrm{sing}})$ and we can thus assume that X is smooth. Then with the choice $X = Y$, we have that c_X is the identity homomorphism $\Omega^r_{X/S} \to \Omega^r_{X/S}$.

In this paper, we are mostly interested in the relative case. Let X/S be a scheme of relative dimension r. A relative fundamental class of X/S will be a class in $\mathrm{Ext}^{-r}(\Omega^r_{X/S}, \mathcal{D}^\bullet_{X/S})$ satisfying certain properties as stated below. The construction of this class is local:

Lemma (A.2) ([AEZ78, Lem. II.1]). *Let S be a noetherian scheme and let $X \to S$ be equidimensional of relative dimension r. Then*

$$\mathrm{Ext}^{-r}(\mathcal{F}, \mathcal{D}^\bullet_{X/S}) = \Gamma\big(X, \mathcal{E}xt^{-r}(\mathcal{F}, \mathcal{D}^\bullet_{X/S})\big)$$

for every $\mathcal{O}_X$-module $\mathcal{F}$.

We will use the following duality isomorphism which is a special case of Theorem (A.1):

Proposition (A.3) ([EZ78, Ch. IV, Prop. 2]). *Let $X \to S$ be equidimensional of relative dimension r and let $Y \to S$ be a smooth morphism of relative dimension r. Let $f : X \to Y$ be a finite S-morphism. Then there is a canonical isomorphism*

$$T_f \,:\, \mathrm{Ext}^{-r}(\Omega^r_{X/S}, \mathcal{D}^\bullet_{X/S}) \xrightarrow{\cong} \mathrm{Hom}(f_*\Omega^r_{X/S}, \Omega^r_{Y/S}).$$

Corollary (A.4) ([EZ78, Ch. IV, Prop. 4]). *Let $X \to S$ be equidimensional of relative dimension r and let $Y \to S$ be a smooth morphism of relative dimension r. Let $f : X \to Y$ be a finite S-morphism. Let $U \subseteq Y$ be a schematically dense open subset. Then the canonical homomorphism*

$$\mathrm{Ext}^{-r}(\Omega^r_{X/S}, \mathcal{D}^\bullet_{X/S}) \to \mathrm{Ext}^{-r}(\Omega^r_{f^{-1}(U)/S}, \mathcal{D}^\bullet_{f^{-1}(U)/S})$$

is injective.

Recall that if $f : X \to Y$ is a finite and flat morphism, then $f_*\mathcal{O}_X$ is locally free and there is a trace homomorphism $f_*\mathcal{O}_X \to \mathcal{O}_Y$. By tensoring with $\Omega^r_{Y/S}$ we obtain the homomorphism

$$\mathrm{tr}(f) \,:\, f_*f^*\Omega^r_{Y/S} \to \Omega^r_{Y/S}.$$

Definition (A.5) ([EZ78, Ch. IV, Def. 2]). Let S be reduced and let $X \to S$ be equidimensional of relative dimension r. We say that a class $c \in \mathrm{Ext}^{-r}(\Omega^r_{X/S}, \mathcal{D}^\bullet_{X/S})$ satisfies the *property of the trace*, if for every open subset $U \subseteq X$, every smooth morphism $Y \to S$ of dimension r and every finite and flat morphism $f : U \to Y$, we have that the composition of $f_*f^*\Omega^r_{Y/S} \to f_*\Omega^r_{X/S}$ and $T_f(c)$ is the trace $\mathrm{tr}(f)$.

Proposition (A.6) ([AEZ78, Prop. II.3.1]). *Let S be reduced and let $X \to S$ be equidimensional of relative dimension r. There is at most one class*

$$c \in \mathrm{Ext}^{-r}(\Omega^r_{X/S}, \mathcal{D}^\bullet_{X/S})$$

satisfying the property of the trace.

Proof. The question is local on X by Lemma (A.2). We can thus assume that there is a closed immersion $j : X \to \mathbb{A}^n$. Let $\varphi : \mathbb{A}^n \to \mathbb{A}^r$ be a linear morphism such that the composition $f : X \to \mathbb{A}^r$ is generically finite. Note that as S is reduced, we have that f is generically flat. The property of the trace determines c over the image of $j^*\varphi^*\Omega^r_{\mathbb{A}^r/S} \to j^*\Omega^r_{\mathbb{A}^n/S} \to \Omega^r_{X/S}$. Let x be a generic point of X, the images of $\varphi^*\Omega^r_{\mathbb{A}^r/S} \to \Omega^r_{\mathbb{A}^n/S}$ for every φ such that $j \circ \varphi$ is quasi-finite at x, generates $\Omega^r_{\mathbb{A}^n/S}$ in a neighborhood of x. For details, see [AEZ78, *loc. cit.*]. $\square$

Definition (A.7). Let S be reduced and let $X \to S$ be equidimensional of relative dimension r. The unique class $c_{X/S} \in \operatorname{Ext}^{-r}(\Omega^r_{X/S}, \mathcal{D}^\bullet_{X/S})$ satisfying the property of the trace, if it exists, is called the relative fundamental class of X/S.

The fundamental class c_X for a scheme X/k discussed above is the relative fundamental class $c_{X/k}$, cf. [EZ78, Ch. III, Cor.].

Remark **(A.8).** Let S be reduced and let $X \to S$ be equidimensional of relative dimension r. If X has irreducible components X_i with multiplicities m_i then it follows that $c_X = \sum_i m_i c_{X_i}$. In fact, if $f : X \to Y$ is a finite morphism, then $\operatorname{tr}(f) = \sum_i m_i \operatorname{tr}(f|_{X_i})$ at the generic points of Y where all involved maps are flat.

The relative fundamental class exists in the following cases:

(i) S normal and X/S equidimensional of dimension r [EZ78, Ch. IV, No. 3].
(ii) S reduced and X/S flat [EZ78, Ch. IV, No. 4].

If S is not reduced things are slightly more complicated. We assume that X/S is flat or at least of finite Tor-dimension. If S is without embedded components, then the property of the trace as stated in Definition (A.5) is enough to ensure uniqueness. In fact, if $U \subseteq X$ is an open subset, Y/S is smooth of relative dimension r, and $f : U \to Y$ is a quasi-finite morphism, then f is generically flat and finite. In general, the property of the trace should be generalized to include morphisms $f : U \to Y$ which are finite and of finite Tor-dimension but not necessarily flat. The trace of such a morphism is defined by the alternating sum of the traces of a flat resolution, cf. [AEZ78, Ch. II]. The main result of [AEZ78] is that for any locally noetherian scheme S of characteristic zero and X/S of finite type and finite Tor-dimension, there exists a relative fundamental class of X/S.

Appendix B. Schematically dominant families

Recall that a family of morphisms $u_\lambda : Z_\lambda \to X$ is schematically dominant if the intersection of the kernels of $\mathcal{O}_X \to (u_\lambda)_*\mathcal{O}_{Z_\lambda}$ is zero [EGA$_{\mathrm{IV}}$, 11.10]. The important fact is that a morphism from X is determined on $\{Z_\lambda\}$, i.e., if Y is a separated scheme, then

$$\operatorname{Hom}(X, Y) \to \prod_\lambda \operatorname{Hom}(Z_\lambda, Y)$$

is injective. In this section we show that if $X \to Y$ is a smooth morphism, then the family of all subschemes $Z_\lambda \hookrightarrow X$ which are étale over Y, is schematically dominant.

Lemma (B.1). *Let S and X be affine schemes and $X \to S$ a smooth morphism. Let $f \in \Gamma(X)$. Then there exists a locally closed subscheme $j : Z \hookrightarrow X$ such that $Z \hookrightarrow X \to S$ is étale and $j^*(f) \in \Gamma(Z)$ is non-zero.*

Proof. Let $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. By a standard limit argument, we can assume that A is noetherian. Let $x \in X$ be a point such that f is not zero in $\mathcal{O}_{X,x}$. Let $s \in S$ be the image of x. Let $\mathfrak{p} \subseteq A$ be the prime ideal corresponding to s.

By Krull's intersection theorem there is an integer $n \geq 0$ such that $f \in \mathfrak{p}^n B_\mathfrak{p}$ but $f \notin \mathfrak{p}^{n+1} B_\mathfrak{p}$. Consider the $k(\mathfrak{p})$-module $M = \mathfrak{p}^n A_\mathfrak{p}/\mathfrak{p}^{n+1} A_\mathfrak{p} \otimes_{k(\mathfrak{p})} B \otimes_A k(\mathfrak{p})$. By flatness, this is the submodule $\mathfrak{p}^n B_\mathfrak{p}/\mathfrak{p}^{n+1} B_\mathfrak{p}$ of $B_\mathfrak{p}/\mathfrak{p}^{n+1} B_\mathfrak{p}$. Choose a basis for $\mathfrak{p}^n A_\mathfrak{p}/\mathfrak{p}^{n+1} A_\mathfrak{p}$ and let $g_1, g_2, \dots, g_k \in B \otimes_A k(\mathfrak{p})$ be the coefficients in this basis of the image of f in M. As f is not zero in M, there is at least one non-zero g_i and we let $g = g_i$.

Let $U \subseteq X$ be an open subset such that $U \cap X_s = (X_s)_g$. Choose a closed point $x \in (X_s)_g$ such that $k(s) \to k(x)$ is separable. There is then by [EGA$_{\mathrm{IV}}$, Cor. 17.16.3] a locally closed affine subscheme $Z = \mathrm{Spec}(C) \hookrightarrow U$, containing the point x, such that $Z \to X \to S$ is étale. In particular we have that g is invertible in $C \otimes_A k(\mathfrak{p})$. It follows that the image of f in $\mathfrak{p}^n C_\mathfrak{p}/\mathfrak{p}^{n+1} C_\mathfrak{p}$ is non-zero. As C is a flat A-algebra, this implies that the image of f in $C_\mathfrak{p}/\mathfrak{p}^{n+1} C_\mathfrak{p}$ is non-zero. In particular, the image of f in C is non-zero. □

Proposition (B.2). *Let S be a scheme and let $X \to S$ be a smooth morphism. Then the family of all subschemes $Z_\lambda \hookrightarrow X$ which are étale over Y, is schematically dominant.*

Proof. It is enough to show that the family is schematically dominant when X and S are affine. Let $f, g \in \Gamma(X, \mathcal{O}_X)$ such that f is non-zero in $\Gamma(X_g, \mathcal{O}_X)$. The above lemma gives a locally closed subscheme $X_{f,g} \hookrightarrow X_g$ such that $X_{f,g} \to S$ is étale and such that the pull-back of f to $X_{f,g}$ is non-zero. It follows that the family $(X_{f,g} \hookrightarrow X)$ is schematically dominant. □

Corollary (B.3). *Let S be a scheme, $S' \to S$ a smooth morphism and $B' \to S'$ a flat morphism, locally of finite presentation. Then there is a family of locally closed subschemes $S'_\lambda \hookrightarrow S'$ such that $S'_\lambda \to S$ is étale and such that $(S'_\lambda \times'_S B' \hookrightarrow B')$ is schematically dominant.*

Proof. Take a schematically dominant family $(S'_\lambda \hookrightarrow S')$ as in the proposition. Then the pull-back family $(S'_\lambda \times'_S B' \hookrightarrow B')$ is schematically dominant as well [EGA$_{\mathrm{IV}}$, Thm. 11.10.5]. □

References

[AB57] Maurice Auslander and David A. Buchsbaum, *Homological dimension in local rings*, Trans. Amer. Math. Soc. **85** (1957), 390–405.

[AB69] A. Andreotti and E. Bombieri, *Sugli omeomorfismi delle varietà algebriche*, Ann. Scuola Norm. Sup Pisa (3) **23** (1969), 431–450.

[AEZ78] B. Angéniol and F. El Zein, *Appendice: La classe fondamentale rélative d'un cycle.*, Bull. Soc. Math. Fr., Suppl., Mém. (1978), no. 58, 67–93.

[AK06] Valery Alexeev and Allen Knutson, *Complete moduli spaces of branchvarieties*, Preprint, Feb 2006, arXiv:math/0602626.

[Ang80] B. Angéniol, *Schéma de chow*, Thèse, Orsay, Paris VI, 1980.

[Art74] M. Artin, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165–189.

[Bar75] Daniel Barlet, *Espace analytique réduit des cycles analytiques complexes compacts d'un espace analytique complexe de dimension finie*, Fonctions de plusieurs variables complexes, II (Sém. François Norguet, 1974–1975), Springer, Berlin, 1975, pp. 1–158. Lecture Notes in Math., Vol. 482.

[Bou61] N. Bourbaki, *Éléments de mathématique. Fascicule XXVII. Algèbre commutative. Chapitre 1: Modules plats. Chapitre 2: Localisation*, Actualités Scientifiques et Industrielles, No. 1290, Herman, Paris, 1961.

[CW37] Wei-Liang Chow and B. L. van der Waerden, *Zur algebraischen Geometrie. IX*, Math. Ann. **113** (1937), no. 1, 692–704.

[Del73] Pierre Deligne, *Cohomologie à supports propres*, Exposé XVII of SGA 4, Théorie des topos et cohomologie étale des schémas. Tome 3, Springer-Verlag, Berlin, 1973, pp. 250–480. Lecture Notes in Math., Vol. 305.

[EGA_{I}] A. Grothendieck, *Éléments de géométrie algébrique. I. Le langage des schémas*, second ed., Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 166, Springer-Verlag, Berlin, 1971.

[EGA_{II}] ———, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 8, 222.

[EGA_{III}] ———, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents*, Inst. Hautes Études Sci. Publ. Math. (1961, 1963), nos. 11, 17.

[EGA_{IV}] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas*, Inst. Hautes Études Sci. Publ. Math. (1964-67), nos. 20, 24, 28, 32.

[EZ78] Fouad El Zein, *Complexe dualisant et applications à la classe fondamentale d'un cycle*, Bull. Soc. Math. France Mém. (1978), no. 58, 66.

[Fer03] Daniel Ferrand, *Conducteur, descente et pincement*, Bull. Soc. Math. France **131** (2003), no. 4, 553–585.

[FGA] A. Grothendieck, *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.]*, Secrétariat mathématique, Paris, 1962.

[Fog68] John Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math **90** (1968), 511–521.

[Fog69] ———, *Truncated Hilbert functors*, J. Reine Angew. Math. **234** (1969), 65–88.

[Fog71] ———, *Universal Weil divisors*, J. Reine Angew. Math. **250** (1971), 141–152.

[Ful98] William Fulton, *Intersection theory*, second ed., vol. 2, Springer-Verlag, Berlin, 1998.

[GIT] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994.

[GKZ94] I. M. Gel′fand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1994.

[GS71] Alexander Grothendieck and Hamet Seydi, *Platitude d'une adhérence schématique et lemme de Hironaka généralisé*, Manuscripta Math. **5** (1971), 323–339.

[Gue96] Lucio Guerra, *A universal property of the Cayley-Chow space of algebraic cycles*, Rend. Sem. Mat. Univ. Padova **95** (1996), 127–142.

[Har66] Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966.

[Har75] ———, *On the De Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. (1975), no. 45, 5–99.

[Høn04] M. Hønsen, *A compact moduli space for cohen-macaulay curves in projective space*, Phd. thesis, MIT, 2004.

[Høn05] ———, *Compactifying locally cohen-macaulay projective curves*, Phd. thesis, KTH, 2005.

[Hoy66] William L. Hoyt, *On the Chow bunches for different projective embeddings of a complete variety*, Amer. J. Math. **88** (1966), 273–278.

[KM76] Finn Faye Knudsen and David Mumford, *The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"*, Math. Scand. **39** (1976), no. 1, 19–55.

[Knu71] Donald Knutson, *Algebraic spaces*, Springer-Verlag, Berlin, 1971, Lecture Notes in Mathematics, Vol. 203.

[Kol95] János Kollár, *Flatness criteria*, J. Algebra **175** (1995), no. 2, 715–727.

[Kol96] ———, *Rational curves on algebraic varieties*, vol. 32, Springer-Verlag, Berlin, 1996.

[Lie06] Max Lieblich, *Remarks on the stack of coherent algebras*, Int. Math. Res. Not. (2006), Art. ID 75273, 12.

[LMB00] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 39, Springer-Verlag, Berlin, 2000.

[Lüt93] W. Lütkebohmert, *On compactification of schemes*, Manuscripta Math. **80** (1993), no. 1, 95–111.

[Man80] Mirella Manaresi, *Some properties of weakly normal varieties*, Nagoya Math. J. **77** (1980), 61–74.

[Nag55] Masayoshi Nagata, *On the normality of the Chow variety of positive* 0*-cycles of degree* m *in an algebraic variety*, Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 165–176.

[Nag62] ———, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **2** (1962), 1–10.

[Rao71] Jean-Claude Raoult, *Compactification des espaces algébriques normaux*, C. R. Acad. Sci. Paris Sér. A-B **273** (1971), A766–A767.

[Rao74] ———, *Compactification des espaces algébriques*, C. R. Acad. Sci. Paris Sér. A **278** (1974), 867–869.

[RG71] Michel Raynaud and Laurent Gruson, *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89.

[Ryd07] David Rydh, *Submersions and effective descent of étale morphisms*, Preprint, Oct 2007, arXiv:0710.2488v2.

[Ryd08a] ———, *Families of zero cycles and divided powers: I. Representability*, Preprint, Mar 2008, arXiv:0803.0618v1.

[Ryd08b] ———, *Families of zero cycles and divided powers: II. The universal family*, Draft, 2008.

[Ryd08c] ———, *Hilbert and Chow schemes of points, symmetric products and divided powers*, In preparation, 2008.

[Ryd08d] ———, *Representability of Hilbert schemes and Hilbert stacks of points*, Preprint, Feb 2008, arXiv:0802.3807v1.

[Sam55] P. Samuel, *Méthodes d'algèbre abstraite en géométrie algébrique*, Springer-Verlag, Berlin, 1955.

[SGA_1] A. Grothendieck (ed.), *Revêtements étales et groupe fondamental*, Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud, Lecture Notes in Mathematics, Vol. 224.

[SGA_6] A. Grothendieck (ed.), *Théorie des intersections et théorème de Riemann-Roch*, Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre, Lecture Notes in Mathematics, Vol. 225.

[SV00] Andrei Suslin and Vladimir Voevodsky, *Relative cycles and Chow sheaves*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 10–86.

[Swa80] Richard G. Swan, *On seminormality*, J. Algebra **67** (1980), no. 1, 210–229.

[Tra70] Carlo Traverso, *Seminormality and Picard group*, Ann. Scuola Norm. Sup. Pisa (3) **24** (1970), 585–595.

[Voe96] V. Voevodsky, *Homology of schemes*, Selecta Math. (N.S.) **2** (1996), no. 1, 111–153.

[Yan83] Hiroshi Yanagihara, *Some results on weakly normal ring extensions*, J. Math. Soc. Japan **35** (1983), no. 4, 649–661.

Department of Mathematics, KTH, 100 44 Stockholm, Sweden
E-mail address: `dary@math.kth.se`