A GENERALIZATION OF LUNA'S FUNDAMENTAL LEMMA FOR STACKS WITH GOOD MODULI SPACES

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ABSTRACT. We generalize Luna's fundamental lemma to smooth morphisms between stacks with good moduli spaces. We also give a precise condition for when it holds for non-smooth morphisms and versions for coherent sheaves and complexes. This generalizes earlier results by Alper, Abramovich–Temkin, Edidin and Nevins.

INTRODUCTION

Let G be a linearly reductive group acting on affine varieties $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Then X and Y admit *good quotients* $\pi_X \colon X \to X /\!\!/ G = \operatorname{Spec} B^G$ and $\pi_Y \colon Y \to Y /\!\!/ G = \operatorname{Spec} A^G$. These are not orbit spaces in general but the closed points of the quotients correspond to closed orbits. Let $f \colon X \to Y$ be a G-equivariant morphism and let $f /\!\!/ G \colon X /\!\!/ G \to Y /\!\!/ G$ be the induced morphism. When f is étale, Luna's fundamental lemma [Lun73, p. 94] gives a criterion for f to be *strongly étale*, that is, $f /\!\!/ G$ is étale and f is the pull-back of $f /\!\!/ G$ along π_Y. An analogous criterion when f is not étale was recently given by Abramovich and Temkin [AT18] when G is diagonalizable.

Stacks with *good moduli spaces* generalize good quotients: a good quotient $X \to X /\!\!/ G$ gives rise to a good moduli space $[X/G] \to X /\!\!/ G$. Luna's fundamental lemma for stacks with good moduli spaces has been generalized for étale morphisms [Alp10, Thm. 6.10] (also see [AHR19, Thm. 3.14]) and for closed regular immersions [Edi16].

The purpose of this article is to give a general formulation of Luna's fundamental lemma for stacks with good moduli spaces that simultaneously generalize all the results mentioned above.

Good moduli spaces. We briefly recall some properties of good moduli spaces (see Section 1). Let $\mathscr{X}$ be an algebraic stack that admits a good moduli space $\pi_X \colon \mathscr{X} \to X$. In particular, X is an algebraic space and π_X is initial among maps to algebraic spaces. The map π_X is universally closed and for every point $x \in |X|$, there is a unique closed point $x_0 \in \pi_X^{-1}(x)$. We say that such a point is *special*.

Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism between algebraic stacks with good moduli spaces X and Y. This induces a commutative diagram

$$
\begin{array}{ccc}
\mathscr{X} & \xrightarrow{\ f\ } & \mathscr{Y} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ g\ } & Y.
\end{array}
$$

We say that f is *strong* if the diagram above is cartesian. We say that f is *special*, if f takes special points to special points.

Definition. Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks and consider the induced morphism $\varphi\colon I_{\mathscr{X}} \to f^*I_{\mathscr{Y}}$ of inertia stacks. We say that f is

  (i) *stabilizer preserving* if φ is an isomorphism;
  (ii) *fiberwise stabilizer preserving at* $y \in |\mathscr{Y}|$ if $\varphi|_{f^{-1}(\mathscr{G}_y)}$ is an isomorphism; and
  (iii) *pointwise stabilizer preserving at* $x \in |\mathscr{X}|$ if $\varphi|_{\mathscr{G}_x}$ is an isomorphism.

A strong morphism is stabilizer preserving and special. We can now state the main theorem.

Theorem A. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism between algebraic stacks that admit good moduli spaces $\pi_X\colon \mathscr{X} \to X$ and $\pi_Y\colon \mathscr{Y} \to Y$. Let $g\colon X \to Y$ denote the induced morphism. Assume that π_X and π_Y have affine diagonals. Further assume either that f is locally of finite presentation or that $\mathscr{X}$ and $\mathscr{Y}$ are locally noetherian. Then f is strong if and only if*

  (i) *f is special,*
  (ii) *f is fiberwise stabilizer preserving at every special point of $\mathscr{Y}$, and*
  (iii) *for every special point $x_0 \in |\mathscr{X}|$, the (ind-)vector bundle $\mathcal{H}^{-1}(\mathbf{L}i_{x_0}^*\mathbb{L}_f)$ is trivial where $i_{x_0}\colon \mathscr{G}_{x_0} \hookrightarrow \mathscr{X}$ denotes the inclusion of the residual gerbe.*

If in addition f has one of the properties:

  (a) *regular, smooth, étale, open immersion,*
  (b) *unramified, closed immersion, locally closed immersion, quasi-regular immersion, Koszul-regular immersion, monomorphism,*
  (c) *flat, syntomic, local complete intersection, quasi-finite, finite, locally of finite presentation;*

then so has g. Moreover

  (1) *If π_Y is a coarse moduli space, then condition (i) is redundant.*
  (2) *If f has reduced fibers, e.g., if f has one of the properties in (a)–(b), then we can replace condition (ii) with the condition that f is pointwise stabilizer preserving at every point of $\mathscr{X}$ above a special point of $\mathscr{Y}$.*
  (3) *If f is flat, then condition (iii) is redundant.*

This generalizes the main theorem of [AT18] in several different directions. Most importantly, we do not require that $\mathscr{X}$ and $\mathscr{Y}$ are quotient stacks, nor that the stabilizer groups are diagonalizable. This answers [AT18, §1.5]. Furthermore, we prove that flatness descends and that the condition on cotangent complexes is redundant for flat morphisms. We also give pointwise

variants of conditions (i)–(iii) and show that if these hold at a special point $x_0 \in \pi_X^{-1}(x)$, then f is strong in a neighborhood of $x \in |X|$ (Theorem 4.2). In particular, if X is quasi-compact and quasi-separated, then it is enough to verify (iii) for closed points x_0. Finally, we also allow $\mathscr{X}$ and $\mathscr{Y}$ to be non-noetherian if we instead assume that f is locally of finite presentation.

Recall that syntomic means flat with fibers that are lci (local complete intersections). We do not assume that lci maps and syntomic maps between noetherian algebraic stacks are of finite type, see Section 6. In the noetherian setting, the notions of regular, Koszul-regular and quasi-regular coincide (see Section 5). A Koszul-regular immersion is an lci immersion.

We also have a similar result for coherent sheaves. This is well-known for vector bundles, see [Alp13, Thm. 10.3] and [DN89, §2]. For coherent sheaves this was proved in the GIT setting by Nevins [Nev08, Thm. 1.2].

Theorem B. *Let $\pi\colon \mathscr{X} \to X$ be a good moduli space. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathscr{X}}$-module of finite presentation. The following are equivalent*

    (i) *$\mathcal{F} = \pi^*\mathcal{G}$ for some $\mathcal{O}_X$-module $\mathcal{G}$ of finite presentation;*
    (ii) *the counit map $\epsilon\colon \pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is bijective; and*
    (iii) *for every special point $x_0 \in |\mathscr{X}|$, the vector bundles $i_{x_0}^*\mathcal{F}$ and $\mathcal{H}^{-1}(\mathbf{L}i_{x_0}^*\mathcal{F})$ are trivial where $i_{x_0}\colon \mathscr{G}_{x_0} \hookrightarrow \mathscr{X}$ denotes the inclusion of the residual gerbe.*

Under these equivalent conditions, $\mathcal{G} = \pi_*\mathcal{F}$ is of finite presentation. Moreover, if $\mathcal{F}$ is flat, i.e., locally free, then it is enough that $i_{x_0}^*\mathcal{F}$ is trivial for every special point x_0 and then $\pi_*\mathcal{F}$ is locally free as well.

See Theorem 1.3 for a more general version of Theorem B. Also see Theorem 1.5 for a version for complexes generalizing [Nev08, Thm. 1.3].

Applications to tame stacks. Recall that a tame stacks is a stack with finite inertia and finite linearly reductive stabilizers [AOV08]. The coarse moduli space of a tame stack is also a good moduli space.

Corollary C. *Let $\mathscr{X}$ be a tame stack with coarse moduli space $\pi\colon \mathscr{X} \to X$. Let $\mathscr{Y}$ be any algebraic stack and let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism. The following are equivalent*

    (i) *f factors through π,*
    (ii) *f factors through the fppf sheafification $\mathscr{X} \to \pi_0\mathscr{X}$,*
    (iii) *the induced morphism $I_f\colon I_{\mathscr{X}} \to I_{\mathscr{Y}}$ of inertia stacks is trivial, i.e., factors through the identity section $\mathscr{Y} \to I_{\mathscr{Y}}$.*
    (iv) *for every field k, every morphism $x\colon \operatorname{Spec} k \to \mathscr{X}$ and every automorphism $\varphi \in \operatorname{Aut}(x)$, the automorphism $f(\varphi) \in \operatorname{Aut}(f \circ x)$ is the identity.*

Under these equivalent conditions, the factorization through π is unique.

Concretely, if for example $\mathscr{Y} = \mathscr{M}_g$ is the stack of smooth genus g curves, then a family of genus g curves $\mathscr{C} \to \mathscr{X}$ comes from family of genus g curves $C \to X$ if, for every point $x \in \mathscr{X}(k)$, the stabilizer group of x acts trivially on $\mathscr{C}_x$. Taking $\mathscr{Y} = BGL_n$ we recover Theorem B for vector bundles on tame stacks. Taking $\mathscr{Y} = BG$ for a flat group scheme $G \to X$ locally of finite presentation, we obtain:

Corollary D. *Let $\mathscr{X}$ be a tame stack with coarse moduli space $\pi\colon \mathscr{X} \to X$. Let $G \to X$ be a flat group scheme locally of finite presentation. Then π^* induces an equivalence of categories between the category of G-torsors $E \to X$ and the category of G-torsors $\mathscr{E} \to \mathscr{X}$ such that $\mathrm{Aut}(x)$ acts trivially on $\mathscr{E}_x$ for every point $x \in \mathscr{X}(k)$.*

The special cases $G = \mathbb{G}_m$, $G = S_n$ and G tame are [Ols12, Props. 6.1, 6.2 and 6.4]. When G is defined over a field, the result is [BB17, Prop. 3.15]. For G étale, the result is also true for non-tame coarse moduli spaces. This follows directly from [Ryd13, Prop. 6.7]. If X is quasi-compact and quasi-separated, then it is enough to verify conditions Theorem B(iii), Corollary C(iv) and Corollary D at closed points.

Necessity of conditions. When f is étale, Luna's fundamental lemma is also true for coarse moduli spaces [Ryd13, Prop. 6.7] and more generally for adequate moduli spaces [AHR19, Thm. 3.14]. Adequate moduli spaces gives a full generalization of GIT in positive characteristic, also allowing for reductive groups that are not linearly reductive.

When f is not étale, these generalizations fail: in [RRZ18, §4.5] there is an example of an action of $G = \mathbb{Z}/p\mathbb{Z}$ on an affine scheme $\mathrm{Spec}\,A$, where $A = \mathbb{F}_p[\epsilon, x]/(\epsilon^2)$, such that

    (i) Theorem A does not hold for $\mathscr{Y} = [\mathrm{Spec}\,A/G]$ and a certain affine smooth morphism f,

    (ii) Theorem B does not hold for $\mathscr{X} = [\mathrm{Spec}\,A/G]$ and a certain torsion line bundle $\mathcal{F}$,

    (iii) Corollary C does not hold for $\mathscr{X} = [\mathrm{Spec}\,A/G]$ and $\mathscr{Y} = B\boldsymbol{\mu}_p$ or $\mathscr{Y} = B\mathbb{G}_m$.

    (iv) Corollary D does not hold for $\mathscr{X} = [\mathrm{Spec}\,A/G]$ and $G = \boldsymbol{\mu}_p$.

If π_Y does not have separated diagonal, then Theorem A fails. Consider the non-separated group scheme $G \to \mathbb{A}^1$ which as a scheme is the affine line with a double origin. We have a surjective homomorphism $(\mathbb{Z}/2\mathbb{Z})_{\mathbb{A}^1} \to G$. Let $\mathbb{G}_m$ act by multiplication on $\mathbb{A}^1$ and let $\mathbb{Z}/2\mathbb{Z}$ and G act trivially on $\mathbb{A}^1$. Then $f\colon [\mathbb{A}^1/(\mathbb{Z}/2\mathbb{Z} \times \mathbb{G}_m)] \to [\mathbb{A}^1/(G \times \mathbb{G}_m)]$ satisfies (i)–(iii) of Theorem A but is not stabilizer preserving. In [AHR19, Thm. 13.1], it is shown under some mild finiteness assumptions that if π_Y has separated diagonal, then it has affine diagonal.

Overview. In Section 1 we recall the notion of a good moduli space and its basic properties, including a variant of Nakayama's lemma (Lemma 1.1). We also prove Theorem B and Theorem A when f is a closed immersion. In Section 2, we prove that conditions (i) and (ii) in Theorem A imply that $\rho\colon \mathscr{X} \to \mathscr{Y} \times_Y X$ is a closed immersion. This is the key to reduce Theorem A to the special case treated in Section 1.

In Section 3 we study condition (iii) and in Section 4 we give the local version of the main theorem (Theorem 4.2) and most of the properties.

In Sections 5–6 we briefly recall the notions of regular immersions and local complete intersections with an emphasis on stacks, maps between non-noetherian schemes/stacks, and maps between noetherian schemes/stacks that are not of finite type. We then prove that the properties quasi-regular, Koszul-regular and lci descend for strong morphisms.

1. Good moduli spaces

In this section we briefly review the theory of good moduli spaces, due to Alper [Alp13, §4] and prove Theorem B.

1.1. **Good moduli spaces.** Let $\mathscr{X}$ be an algebraic stack. We say that $\mathscr{X}$ admits a *good moduli space* if there exists an algebraic space X and a morphism $\pi_X \colon \mathscr{X} \to X$ such that

   (i) π_X is quasi-compact and quasi-separated,
   (ii) π_X is cohomologically affine, i.e., the functor $(\pi_X)_* \colon \mathbf{QCoh}(\mathscr{X}) \to \mathbf{QCoh}(X)$ is exact, and
   (iii) $(\pi_X)_* \mathcal{O}_{\mathscr{X}} = \mathcal{O}_X$.

Uniqueness. The space X, if it exists, is unique [Alp13, Thm. 6.6] (see [AHR19, Thm. 3.12] for the non-noetherian case).

Noetherian and coherence. If $\mathscr{X}$ is noetherian, then so is X and $(\pi_X)_*$ preserves coherence [Alp13, Thm. 4.16 (x)]. If in addition π_X has affine diagonal (e.g., $\mathscr{X}$ has affine diagonal), then π_X is of finite type [AHR20, Thm. A.1].

Quasi-coherent sheaves. The functor π_X^* is fully faithful, that is, the unit map $\mathcal{F} \to (\pi_X)_*(\pi_X)^* \mathcal{F}$ is an isomorphism for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ [Alp13, Prop. 4.5]. This follows from the definitions after choosing a free presentation of $\mathcal{F}$ locally on X.

Base change. If we have a cartesian diagram

$$
\begin{array}{ccc}
\mathscr{X} & \xrightarrow{\ f\ } & \mathscr{Y} \\
{\scriptstyle \pi_X}\big\downarrow & \square & \big\downarrow{\scriptstyle \pi_Y} \\
X & \xrightarrow{\ g\ } & Y
\end{array}
$$

where X and Y are algebraic spaces and π_Y is a good moduli space, then π_X is a good moduli space and push-forward along π_Y commutes with base change: $g^*(\pi_Y)_* = (\pi_X)_* f^*$ [Alp13, Prop. 4.7]. If f is affine, then so is g and $g_* \mathcal{O}_X = (\pi_Y)_* f_* \mathcal{O}_{\mathscr{X}}$.

Purity. The good moduli space morphism $\pi_X \colon \mathscr{X} \to X$ is *pure*, that is, if $\pi_{X'} \colon \mathscr{X}' \to X'$ is the base change of π_X along any map $X' \to X$, then $\mathcal{O}_{X'} \to (\pi_{X'})_* \mathcal{O}_{\mathscr{X}'}$ is injective. This follows from the definition of a good moduli space since $\pi_{X'}$ also is a good moduli space.

Residual gerbes. Let $x \in |\mathscr{X}|$ be a point. If $\mathscr{X}$ is quasi-separated (or equivalently, X is quasi-separated), then the *residual gerbe* $\mathscr{G}_x$ exists and comes along with a quasi-affine monomorphism $i_x \colon \mathscr{G}_x \hookrightarrow \mathscr{X}$ [Ryd11, Thm. B.2]. The residual gerbe $\mathscr{G}_x$ is an fppf gerbe over the *residue field* $\kappa(x)$. When x is a closed point, i_x is a closed immersion corresponding to the maximal ideal of x.

Topological properties. The morphism π_X is universally closed. For every point $x \in |X|$, there is a unique closed point $x_0 \in \pi^{-1}(x)$ [Alp13, Thm. 4.16 (iv)]. We say that such a point is *special*. If $\mathscr{Z} \hookrightarrow \mathscr{X}$ is a closed substack, then $Z := \pi_X(\mathscr{Z})$ is a closed subspace of X and $\mathscr{Z} \to Z$ is a good moduli space. In particular, if x_0 is special, then the residual gerbe $\mathscr{G}_{x_0}$ has good moduli space $\operatorname{Spec} \kappa(x)$. In particular, the residue fields $\kappa(x) = \kappa(x_0)$ coincide.

1.2. **Nakayama's lemma.** Let $\mathscr{X}$ be an algebraic stack and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_{\mathscr{X}}$-module of *finite type*. Nakayama's lemma then implies that $\operatorname{Supp} \mathcal{F}$ is closed and that $x \in \operatorname{Supp} \mathcal{F}$ if and only if $i_x^* \mathcal{F} = 0$. In particular, if $i_x^* \mathcal{F} = 0$, then $\mathcal{F}|_U = 0$ for an open neighborhood U of x. If $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a homomorphism of quasi-coherent $\mathcal{O}_{\mathscr{X}}$-modules and $\mathcal{G}$ is of finite type, then $\operatorname{coker} \varphi$ is of finite type and we conclude that φ is surjective in a neighborhood of x if and only if $i_x^* \varphi$ is surjective.

We have the following version of Nakayama's lemma for stacks with good moduli spaces.

Lemma (1.1). *Let $\mathscr{X}$ be an algebraic stack with good moduli space $\pi \colon \mathscr{X} \to X$. Let $x \in |X|$ be a point and $x_0 \in |\mathscr{X}|$ be the unique special point above x.*
   (i) *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathscr{X}}$-module of finite type. If $i_{x_0}^* \mathcal{F} = 0$, then $\mathcal{F}|_{\pi^{-1}(U)} = 0$ for an open neighborhood U of x.*
   (ii) *Let $\varphi \colon \mathcal{F} \to \mathcal{G}$ be a homomorphism of quasi-coherent $\mathcal{O}_{\mathscr{X}}$-modules where $\mathcal{G}$ is of finite type. If $i_{x_0}^* \varphi$ is surjective, then $\varphi|_{\pi^{-1}(U)}$ is surjective for an open neighborhood U of x.*

Proof. For (i), take $U = X \smallsetminus \pi(\operatorname{Supp} \mathcal{F})$ and for (ii) consider $\operatorname{coker} \varphi$. $\square$

1.3. **Proof of Theorem B.** We will need the following lemma.

Lemma (1.2). *Let $\mathscr{X}$ be an algebraic stack with good moduli space $\pi \colon \mathscr{X} \to X$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathscr{X}}$-module. Let $x_0 \in |\mathscr{X}|$ be a special point and $x = \pi(x_0)$. If $\mathcal{F}$ is of finite type and $i_{x_0}^* \mathcal{F}$ is trivial, then after base change along an étale neighborhood $X' \to X$ of x there exists a surjective homomorphism*
$$\varphi \colon \mathcal{O}_{\mathscr{X}}^{\oplus n} \twoheadrightarrow \mathcal{F}$$
such that $i_{x_0}^* \varphi$ is an isomorphism.

Proof. First assume that X is local henselian with closed point x. By assumption $i_{x_0}^* \mathcal{F} = \mathcal{O}_{\mathscr{G}_{x_0}}^{\oplus n}$. The surjective map $\mathcal{O}_{\mathscr{X}}^{\oplus n} \twoheadrightarrow (i_{x_0})_* \mathcal{O}_{\mathscr{G}_{x_0}}^{\oplus n} = (i_{x_0})_* i_{x_0}^* \mathcal{F}$ lifts to a map $\varphi \colon \mathcal{O}_{\mathscr{X}}^{\oplus n} \to \mathcal{F}$. Indeed, $\Gamma(\mathscr{X}, \mathcal{F}) \to \Gamma(\mathscr{X}, (i_{x_0})_* i_{x_0}^* \mathcal{F})$ is surjective since $\mathscr{X}$ is cohomologically affine. By construction $i_{x_0}^* \varphi$ is an isomorphism. For general X we then obtain, after replacing X with an étale neighborhood, a homomorphism $\varphi \colon \mathcal{O}_{\mathscr{X}}^{\oplus n} \to \mathcal{F}$ such $i_{x_0}^* \varphi$ is an isomorphism.

Then φ is surjective after replacing X with an open neighborhood of x by Lemma 1.1(ii).   □

We now state and prove the following generalization of Theorem B.

Theorem (1.3). *Let $\mathscr{X}$ be an algebraic stack with good moduli space $\pi\colon \mathscr{X} \to X$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathscr{X}}$-module. Let $x \in |X|$ and let $x_0 \in |\mathscr{X}|$ be the unique special point above x. Let $\epsilon\colon \pi^*\pi_*\mathcal{F} \to \mathcal{F}$ denote the counit map.*

  (i) *If $\mathcal{F}$ is of finite type, then ϵ is surjective in a neighborhood of x if and only if $i_{x_0}^*\mathcal{F}$ is a trivial vector bundle. If this is the case, then $\pi_*\mathcal{F}$ is of finite type in a neighborhood of x.*

  (ii) *If $\mathcal{F}$ is of finite presentation, then ϵ is an isomorphism in a neighborhood of x if and only if $i_{x_0}^*\mathcal{F}$ and $\mathcal{H}^{-1}(\mathbf{L}i_{x_0}^*\mathcal{F})$ are trivial vector bundles. If this is the case, then $\pi_*\mathcal{F}$ is of finite presentation in a neighborhood of x.*

  (iii) *If $\mathcal{F}$ is a vector bundle, then the following are equivalent*
      (a) *ϵ is an isomorphism in a neighborhood of x;*
      (b) *ϵ is surjective in a neighborhood of x;*
      (c) *$i_{x_0}^*\mathcal{F}$ is a trivial vector bundle;*
      and under these conditions, $\pi_*\mathcal{F}$ is a vector bundle in a neighborhood of x.

Proof. The questions are local on X so we may assume that X is affine. If ϵ is surjective in a neighborhood of x, then $i_{x_0}^*\epsilon$ is surjective so $i_{x_0}^*\mathcal{F}$ is trivial. Indeed, $i_{x_0}^*\pi^*\pi_*\mathcal{F} = i_x^*\pi_*\mathcal{F} \otimes_{\kappa(x)} \mathcal{O}_{\mathscr{G}_{x_0}}$ is always trivial. Conversely, if $\mathcal{F}$ is of finite type and $i_{x_0}^*\mathcal{F}$ is trivial, then by Lemma 1.2 there is, after replacing X with a neighborhood, a surjective homomorphism $\varphi\colon \mathcal{O}_{\mathscr{X}}^{\oplus n} \to \mathcal{F}$. It follows that ϵ is surjective. Since π_* is exact we also get a surjective homomorphism $\pi_*\varphi\colon \mathcal{O}_X^{\oplus n} \to \pi_*\mathcal{F}$ so $\pi_*\mathcal{F}$ is of finite type.

If $\mathcal{F}$ is of finite presentation and the conditions in (i) are satisfied, then we can find a surjective homomorphism $\varphi\colon \mathcal{O}_{\mathscr{X}}^{\oplus n} \to \mathcal{F}$ such that $i_{x_0}^*\varphi$ is an isomorphism. Let $\mathcal{K} = \ker(\varphi)$ which is of finite type. We note that $\mathcal{H}^{-1}(\mathbf{L}i_{x_0}^*\mathcal{F}) = i_{x_0}^*\mathcal{K}$ and that $\epsilon_{\mathcal{K}}\colon \pi^*\pi_*\mathcal{K} \to \mathcal{K}$ is surjective if and only if $\epsilon_{\mathcal{F}}\colon \pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is an isomorphism. The first claim of (ii) thus follows from (i) applied to $\mathcal{K}$. The last claim of (ii) also follows since $0 \to \pi_*\mathcal{K} \to \mathcal{O}_X^{\oplus n} \to \pi_*\mathcal{F} \to 0$ is exact.

If $\mathcal{F}$ is a vector bundle then (a) $\implies$ (b) $\iff$ (c) by (i). If (c) holds, then $\varphi\colon \mathcal{O}_{\mathscr{X}}^{\oplus n} \to \mathcal{F}$ is a surjective homomorphism of vector bundles of rank n, hence an isomorphism. It follows that $\pi_*\mathcal{F} = \mathcal{O}_X^{\oplus n}$ is a vector bundle of rank n. Then ϵ is a surjection of vector bundles of rank n, hence an isomorphism.   □

Proof of Theorem B. If $\mathcal{F} = \pi^*\mathcal{G}$, then the counit map $\epsilon\colon \pi^*\pi_*\pi^*\mathcal{G} \to \pi^*\mathcal{G}$ is bijective since the unit map $\eta\colon \mathcal{G} \to \pi_*\pi^*\mathcal{G}$ is bijective. The remaining claims follows from Theorem 1.3(ii) and (iii) applied to all points $x \in |X|$.   □

As a consequence, we can prove Theorem A when f is a closed immersion.

Corollary (1.4). *Let $f\colon \mathscr{X} \hookrightarrow \mathscr{Y}$ be a closed immersion of stacks with good moduli spaces $\pi_X\colon \mathscr{X} \to X$ and $\pi_Y\colon \mathscr{Y} \to Y$. Then the induced map $g\colon X \hookrightarrow Y$ is a closed immersion. If $\mathcal{I}$ denotes the ideal sheaf of f, then*

g is given by the ideal sheaf $\mathcal{J} = (\pi_Y)_*\mathcal{I}$. Let $\epsilon\colon (\pi_Y)^*(\pi_Y)_*\mathcal{I} \to \mathcal{I}$ denote the counit map. Let $x \in |X|$ be a point and let $x_0 \in |\mathscr{X}|$ be the unique special point above x. If f is finitely presented at x_0, then the following are equivalent:

    (i) f is strong in a neighborhood of x;
    (ii) ϵ is surjective in a neighborhood of x; and
    (iii) $i_{x_0}^*\mathcal{I}$ is a trivial vector bundle.

When the equivalent conditions hold, then g is finitely presented at x.

Proof. (i) $\iff$ (ii) follows immediately since $(\pi_Y)^{-1}(X)$ is given by the ideal which is the image of ϵ. (ii) $\iff$ (iii) follows from Theorem 1.3(i) applied to $\mathcal{F} = \mathcal{I}$ which is of finite type. The theorem also shows that $\mathcal{J} = (\pi_Y)_*\mathcal{I}$ is of finite type, that is, g is finitely presented. $\qquad\square$

Note that since f is a closed immersion, the naive cotangent complex $\tau^{\geq -1}\mathbb{L}_f = \mathcal{I}/\mathcal{I}^2[1]$ is concentrated in degree -1 so $i_{x_0}^*\mathcal{I} = \mathcal{H}^{-1}(\mathbf{L}i_{x_0}^*\mathbb{L}_f)$.

1.4. Descent for complexes.

We also have a version for Theorem B for complexes generalizing [Nev08, Thm. 1.3]. Recall that a complex $\mathcal{F}^\bullet \in \mathbf{D}_{\mathrm{qc}}(\mathscr{X})$ is m-pseudo-coherent if locally there is a perfect complex $\mathcal{P}^\bullet$ and a morphism $\varphi\colon \mathcal{P}^\bullet \to \mathcal{F}^\bullet$ such that $\mathcal{H}^d(\varphi)$ is an isomorphism for $d > m$ and a surjection for $d = m$ [SP, 08CA]. If $\mathscr{X}$ is noetherian, this simply means that $\mathcal{F}^\bullet$ is bounded above and has coherent cohomology in degrees $\geq m$ [SP, 08E8]. We say that $\mathcal{F}^\bullet$ is pseudo-coherent if it is m-pseudo-coherent for all m. In the noetherian case this simply means $\mathcal{F}^\bullet \in \mathbf{D}_{\mathrm{Coh}}^-(\mathscr{X})$.

Theorem (1.5). *Let $\mathscr{X}$ be an algebraic stack with good moduli space $\pi\colon \mathscr{X} \to X$. Assume that π has affine diagonal. Let $\mathcal{F}^\bullet \in \mathbf{D}_{\mathrm{qc}}(\mathscr{X})$ be a complex of lisse-étale $\mathcal{O}_{\mathscr{X}}$-modules with quasi-coherent cohomology. Let $x \in |X|$ and let $x_0 \in |\mathscr{X}|$ be the unique special point above x. Let $\epsilon\colon \mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{F}^\bullet \to \mathcal{F}^\bullet$ denote the counit map. Let m be an integer. The following are equivalent:*

    (i) *$\mathcal{F}^\bullet$ is m-pseudo-coherent at x_0 and $\mathcal{H}^d(\mathbf{L}i_{x_0}^*\mathcal{F}^\bullet)$ is trivial for every $d \geq m$.*
    (ii) *$\mathbf{R}\pi_*\mathcal{F}^\bullet$ is m-pseudo-coherent and $\mathcal{H}^d(\epsilon)$ is an isomorphism for $d > m$ and surjective for $d = m$ after restricting to an open neighborhood of x.*

Proof. Since the good moduli space map π has affine diagonal π_* is of cohomological dimension zero so that $\mathbf{R}\pi_*$ is t-exact [Alp13, Rmk. 3.5].

That (ii) $\implies$ (i) is immediate. For the converse, let k denote the largest integer such that $\mathcal{H}^k(\mathbf{L}i_{x_0}^*\mathcal{F}^\bullet) \neq 0$. Then there is an open neighborhood of x_0 such that $\mathcal{F}^\bullet$ is acyclic in degrees $\geq \max(m, k+1)$ [HR17, Lem. 4.8]. After restricting to an open neighborhood of x, we may thus assume that $\mathcal{F}^\bullet$ is acyclic in degrees $\geq \max(m, k+1)$. If $k < m$, then there is nothing to prove. If $k \geq m$, then $\mathcal{H}^k(\mathcal{F}^\bullet)$ is of finite type and commutes with base change. By Lemma 1.2 we can, after passing to an open neighborhood of x, find a homomorphism $\varphi\colon \mathcal{O}_{\mathscr{X}}^{\oplus n}[-k] \to \mathcal{F}^\bullet$ such that $\mathcal{H}^k(\varphi)$ is surjective and $\mathcal{H}^k(\mathbf{L}i_{x_0}^*\varphi)$ is an isomorphism. In particular, $\mathcal{H}^k(\epsilon)$ is surjective, $\mathcal{H}^k(\mathbf{R}\pi_*\mathcal{F}^\bullet)$ is of finite type and $\mathbf{R}\pi_*\mathcal{F}^\bullet$ is acyclic in degrees $\geq k+1$. If $k = m$, then we are done. If not, let $\mathcal{G}^\bullet$ be a cone of φ. Then $\mathcal{G}^\bullet$ is m-pseudo-coherent and acyclic in degrees $\geq k$ after passing to an open neighborhood of x. Thus, $\mathcal{H}^k(\epsilon_{\mathcal{F}^\bullet})$

is bijective if $\mathcal{H}^{k-1}(\epsilon_{\mathcal{G}}\bullet)$ is surjective. Moreover, $\mathcal{H}^d(\mathbf{L}i_{x_0}^*\mathcal{G}^\bullet) = \mathcal{H}^d(\mathbf{L}i_{x_0}^*\mathcal{F}^\bullet)$ for all $d \leq k-1$. The result now follows by induction on k. $\qquad\square$

Corollary (1.6). *Let $\mathscr{X}$ be an algebraic stack with good moduli space $\pi\colon \mathscr{X} \to X$. Assume that π has affine diagonal. Let $\mathcal{F}^\bullet \in \mathbf{D}_{\mathrm{qc}}(\mathscr{X})$ be a complex. Then the following are equivalent:*

(i) *$\mathcal{F}^\bullet$ is pseudo-coherent and $\mathcal{H}^d(\mathbf{L}i_{x_0}^*\mathcal{F}^\bullet)$ is trivial for every integer d and every special point $x_0 \in |\mathscr{X}|$.*
(ii) *$\mathbf{R}\pi_*\mathcal{F}^\bullet$ is pseudo-coherent and the counit $\mathbf{L}\pi^*\mathbf{R}\pi_*\mathcal{F}^\bullet \to \mathcal{F}^\bullet$ is a quasi-isomorphism.*

Under the equivalent conditions, $\mathcal{F}^\bullet$ is perfect if and only if $\mathbf{R}\pi_*\mathcal{F}^\bullet$ is perfect. If X is quasi-compact and quasi-separated, then it is enough to consider closed points x_0.

1.5. **Remark on non-quasi-separated spaces.** The following remark clarifies the meaning of i_{x_0} when X is not quasi-separated.

Remark **(1.7).** Let $\pi\colon \mathscr{X} \to X$ be a good moduli space map and let $x_0 \in |\mathscr{X}|$ be a special point with image $x \in |X|$. If X is not quasi-separated, then $\mathscr{X}$ is not quasi-separated and it may happen that there is no residual gerbe $i_{x_0}\colon \mathscr{G}_{x_0} \hookrightarrow \mathscr{X}$. This is merely a notational problem as $\mathscr{X}$ becomes quasi-separated étale-locally on X. Let us say that a *gerbe representative* of x_0 is a stabilizer-preserving morphism $i_{x_0}\colon \mathscr{G} \to \mathscr{X}$ such that $\mathscr{G}$ is a quasi-separated fppf-gerbe over a field k with image $\{x_0\}$. Equivalently, $\mathscr{G}$ is the residual gerbe of the unique closed point of $\mathscr{X} \times_X \operatorname{Spec} k$ for the induced morphism $\operatorname{Spec} k \to X$. When X is quasi-separated, we recover the residual gerbe by taking $\operatorname{Spec}\kappa(x) \to X$.

If $\mathcal{F}$ is a quasi-coherent sheaf of finite type on $\mathscr{X}$, then $i_{x_0}^*\mathcal{F}$ is a vector bundle on $\mathscr{G}$. That this vector bundle vanishes or is trivial does not depend on the choice of gerbe representative i_{x_0}. Similarly, whether $i_{x_0}^*\varphi$ is injective/surjective/bijective does not depend on the choice of i_{x_0}.

If $x_0\colon \operatorname{Spec} k \to \mathscr{X}$ is an ordinary representative of x_0, then $i_{x_0}^*\mathcal{F} = 0$ if and only if $x_0^*\mathcal{F} = 0$ and similarly for injectivity/surjectivity/bijectivity of $i_{x_0}^*\varphi$. The vector bundle $x_0^*\mathcal{F}$ comes with an action of the stabilizer group of x_0. This action is trivial if and only if $i_{x_0}^*\mathcal{F}$ is a trivial vector bundle.

2. Special and stabilizer preserving morphisms

Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism between algebraic stacks that admit good moduli spaces. Recall that f is *special* if it maps special points to special points. If $\mathscr{X}$ and $\mathscr{Y}$ are of finite type over Y, then f is special if and only if f is *weakly saturated* [Alp10, Def. 2.8]. We introduce the following pointwise variants of the conditions in Theorem A:

Definition (2.1). Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism between algebraic stacks with good moduli spaces. Let $x \in |X|$ be a point and let $x_0 \in |\mathscr{X}|$ be the corresponding special point. We say that:

(i) f is (-1)-strong at x if $\operatorname{stab}(x_0) \to \operatorname{stab}(f(x_0))$ is injective;
(ii) f is 0-strong at x if $f(x_0)$ is special and $\pi_X^{-1}(x) \to \mathscr{Y}$ is fiberwise stabilizer preserving at $f(x_0)$;

(iii) f is 1-strong at x if it is 0-strong at x and the (ind-)vector bundle $\mathcal{H}^{-1}(\mathbf{L}i_{x_0}^*\mathbb{L}_f)$ is trivial.

Let $\rho\colon \mathscr{X} \to \mathscr{Y} \times_Y X$ be the natural map and $\rho|_x\colon \mathscr{X} \times_X \operatorname{Spec}\kappa(x) \to \mathscr{Y} \times_Y \operatorname{Spec}\kappa(x)$ its base change. Note that a point $y \in |\mathscr{Y} \times_Y X|$ is special if and only if its image in $|\mathscr{Y}|$ is special. Thus, if $n \in \{-1, 0\}$ then f is n-strong at $x \iff \rho$ is n-strong at $x \iff \rho|_x$ is n-strong at x. Representable morphisms are (-1)-strong and closed immersions are 0-strong. We conclude that f is (-1)-strong (resp. 0-strong) if $\rho|_x$ is representable (resp. a closed immersion).

The main result of this section (Proposition 2.6) gives a reverse implication: if f is 0-strong at x, then ρ is a closed immersion in an open neighborhood of x. We also prove that if f is (-1)-strong at x, then ρ is affine in an open neighborhood of x (Lemma 2.5).

Corollary 1.4 says that if a closed immersion is 1-strong at x, then it is strong in an open neighborhood. The general form of the main theorem (Theorem 4.2) says the same for general f.

We begin with some preparatory lemmas.

Lemma (2.2). *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism between stacks with good moduli spaces and let $y \in |\mathscr{Y}|$ be a special point. Assume that*

(i) *the stabilizer group of y is affine (e.g., Δ_{π_Y} is affine), and*
(ii) *$f^{-1}(y)$ is reduced.*

If f is pointwise stabilizer preserving at every point of $f^{-1}(y)$ then f is fiberwise stabilizer preserving at y and every point of $f^{-1}(y)$ is special.

Proof. The question is local on X so we can assume that X is affine so that $\mathscr{X}$ is cohomologically affine. We can also replace Y with $\operatorname{Spec}\kappa(y)$ and $\mathscr{Y}$ with the residual gerbe $\mathscr{G}_y$ and $\mathscr{X}$ with $f^{-1}(y)$. Then f is representable, because it is pointwise stabilizer preserving. Moreover, f is cohomologically affine, because $\mathscr{X}$ is cohomologically affine and $\mathscr{Y}$ has affine diagonal. It follows that f is affine by Serre's criterion [Alp13, Props. 3.3 and 3.10 (vii)].

Since f is representable and separated, the map between inertia $\varphi\colon I_{\mathscr{X}} \to f^*I_{\mathscr{Y}}$ is a closed immersion. Moreover, φ is surjective, hence a nil-immersion, since f is pointwise stabilizer preserving at every point. Since $\mathscr{Y}$ is a gerbe, $f^*I_{\mathscr{Y}}$ is flat over $\mathscr{X}$, which is reduced. Thus, every (weakly) associated point of $f^*I_{\mathscr{Y}}$ lies over a generic point of $\mathscr{X}$. Over these points, φ is an isomorphism by assumption, so φ is an isomorphism.

The final claim follows from the following lemma which is due to Daniel Bergh.

Lemma (2.3). *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism between quasi-separated algebraic stacks. If $\mathscr{Y}$ is an fppf-gerbe over an algebraic space Y and f is stabilizer preserving, then $\mathscr{X}$ is an fppf-gerbe over an algebraic space X and there is a cartesian diagram:*

$$
\begin{array}{ccc}
\mathscr{X} & \xrightarrow{\;f\;} & \mathscr{Y} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;g\;} & Y.
\end{array}
$$

Proof. Since $I_{\mathscr{X}} = f^* I_{\mathscr{Y}}$ is flat and of finite presentation, $\mathscr{X}$ is a gerbe over an algebraic space X. The structure map $\mathscr{X} \to X$ is initial among maps to algebraic spaces so we have a commutative diagram as above. The map $\mathscr{X} \to \mathscr{Y} \times_Y X$ is a faithfully flat quasi-compact monomorphism, hence an isomorphism. $\qquad\square$

Remark **(2.4).** In Lemma 2.2, it is not enough that f is pointwise stabilizer preserving at every *special* point of $f^{-1}(y)$. Indeed, consider $\mathbb{G}_m$ acting by multiplication on $\mathbb{A}^1$. Then $f \colon [\mathbb{A}^1/\mathbb{G}_m] \to B\mathbb{G}_m$ is smooth and pointwise stabilizer preserving at the unique special point but not stabilizer preserving. It is also not enough that f is pointwise stabilizer preserving at every point of $\pi_X^{-1}(x)$ to deduce that $\pi_X^{-1}(x) \to \mathscr{Y}$ is fiberwise stabilizer preserving at y. Indeed, if $f \colon [\mathbb{A}^1/\boldsymbol{\mu}_2] \to B\boldsymbol{\mu}_2$ where $\boldsymbol{\mu}_2$ acts by multiplication, then f is smooth and pointwise stabilizer preserving at the origin but $\pi_X^{-1}(0) = \left[\left(\operatorname{Spec} k[x]/(x^2)\right)/\boldsymbol{\mu}_2\right] \to B\boldsymbol{\mu}_2$ is not stabilizer preserving. The latter map also shows that it is necessary that $f^{-1}(y)$ is reduced in Lemma 2.2.

Lemma (2.5). *Let $f \colon \mathscr{X} \to \mathscr{Y}$ be a morphism between algebraic stacks with good moduli spaces. Consider the induced map $\rho \colon \mathscr{X} \to \mathscr{Y} \times_Y X$.*

   (i) *If either $\mathscr{X}$ is noetherian or f is locally of finite type, then ρ is of finite type.*
   (ii) *If Δ_{π_X} and Δ_{π_Y} are affine and f is (-1)-strong at $x \in |X|$, then $\rho|_U \colon \mathscr{X} \times_X U \to \mathscr{Y} \times_Y U$ is affine for some open neighborhood U of x.*
   (iii) *f is 0-strong at $x \in |X|$ if and only if $w_0 := \rho(x_0)$ is special, and $\rho^{-1}(\mathscr{G}_{w_0}) \to \mathscr{G}_{w_0}$ is an isomorphism.*

Proof. Recall that if $\mathscr{X}$ is noetherian, then $\mathscr{X} \to X$ is of finite type. In either case, we thus deduce that ρ is locally of finite type. Since π_X and π_Y are quasi-compact and quasi-separated, so is ρ.

For (ii), we may assume that X and Y are affine. Then $\mathscr{X}$ is cohomologically affine and $\Delta_{\mathscr{Y}}$ is affine and the result follows from [AHR19, Prop. 12.5 (1)] when x is closed. For non-closed x, we can instead conclude that $\mathscr{X} \times_X \operatorname{Spec} \mathcal{O}_{X,x} \to \mathscr{Y}$ is affine and then that $\mathscr{X} \times_X U \to \mathscr{Y}$ is affine for some open neighborhood U by [Ryd15, Thm. C].

For (iii), first note that f is 0-strong at x if and only if $\rho|_x$ is 0-strong at x. We can thus assume that $X = Y = \operatorname{Spec} \kappa(x)$ and that $\rho = f$. Then $x_0 \in |\mathscr{X}|$ and $w_0 \in |\mathscr{Y}|$ are closed. Let $\mathscr{X}_{w_0} = f^{-1}(\mathscr{G}_{w_0})$. Since $\mathscr{X}_{w_0} \hookrightarrow \mathscr{X}$ is a closed immersion, the good moduli space of $\mathscr{X}_{w_0}$ coincides with $X = Y = \operatorname{Spec} \kappa(x)$. If $\mathscr{X}_{w_0} \to \mathscr{G}_{w_0}$ is stabilizer preserving, then $\mathscr{X}_{w_0}$ is a gerbe over $\operatorname{Spec} \kappa(x)$ and $\mathscr{X}_{w_0} \to \mathscr{G}_{w_0}$ is an isomorphism (Lemma 2.3). The converse is obvious. $\qquad\square$

The following proposition is a key result in the paper that is used to reduce the main theorem from arbitrary maps to closed immersions so that we can use the results of the previous section.

Proposition (2.6). *Let $f \colon \mathscr{X} \to \mathscr{Y}$ be a morphism between algebraic stacks with good moduli spaces. Consider the map $\rho \colon \mathscr{X} \to \mathscr{Y} \times_Y X$. Assume that π_X and π_Y have affine diagonals and that either $\mathscr{X}$ is noetherian or f is*

locally of finite type. Let $U \subseteq |X|$ be the set of points $x \in |X|$ such that f is 0-strong at x. Then

    (i) *U is open in X.*
    (ii) *$\rho|_U \colon \mathscr{X} \times_X U \to \mathscr{Y} \times_Y U$ is a closed immersion.*

Proof. Let $x \in U$. After replacing X with an open neighborhood of x, we can assume that ρ is affine and of finite type (Lemma 2.5). We can also replace $\mathscr{Y}$ with $\mathscr{Y} \times_Y X$ and assume that $X = Y$. Let $y_0 = f(x_0)$ which is special. Then $f^{-1}(\mathscr{G}_{y_0}) \to \mathscr{G}_{y_0}$ is an isomorphism (Lemma 2.5). In particular, f is quasi-finite at x_0. After replacing $X = Y$ with an open neighborhood, we can thus assume that f is quasi-finite and affine.

By Zariski's main theorem [Ryd16, Thm. 8.1], f factors as an open immersion $j \colon \mathscr{X} \to \mathscr{Y}'$ followed by a finite morphism $g \colon \mathscr{Y}' \to \mathscr{Y}$. Note that $\mathscr{Y}'$ has a good moduli space Y' which is affine over Y. We may thus replace $\mathscr{Y}'$ with $\mathscr{Y}' \times_{Y'} X$ so that $X = Y' = Y$. Then $j(x_0)$ is the unique special point above x so $j|_x$ is an isomorphism. After replacing $X = Y$ with the open neighborhood $Y' \smallsetminus \pi_{Y'}(\mathscr{Y}' \smallsetminus \mathscr{X})$ we can thus assume that $\mathscr{X} = \mathscr{Y}'$ so that f is finite.

Let us finally prove that $f|_U$ is a closed immersion for an open neighborhood U of x. Since $\mathcal{O}_{\mathscr{Y}} \to f_* \mathcal{O}_{\mathscr{X}}$ is finite, the cokernel $\mathcal{F}$ is of finite type. Since $f^{-1}(\mathscr{G}_{y_0}) \to \mathscr{G}_{y_0}$ is an isomorphism, we have that $y_0 \notin \operatorname{Supp} \mathcal{F}$. We now take U to be the complement of the closed subset $\pi_Y(\operatorname{Supp} \mathcal{F})$. $\qquad\square$

Remark **(2.7).** If f is special at x_0 and *pointwise* stabilizer preserving at every point in the fiber $f^{-1}(f(x_0))$, then a similar argument shows that $\rho|_U$ is finite.

Remark **(2.8).** A consequence of Proposition 2.6 is that if f is 0-strong at x, then f is stabilizer preserving at x_0, that is, f is stabilizer preserving after restricting to an open neighborhood U of x.

3. Cotangent complexes

Let $\mathscr{X}$ be an algebraic stack with a good moduli space and let $x \colon \operatorname{Spec} k \to \mathscr{X}$ be a point. Coherent sheaves on the residual gerbe $\mathscr{G}_x$ are vector bundles, that is, locally free sheaves of finite rank. Quasi-coherent sheaves on $\mathscr{G}_x$ are ind-vector bundles, that is, union of vector bundles. We say that an ind-vector bundle is *trivial* if it is free as a quasi-coherent sheaf. An ind-vector bundle $\mathcal{F}$ on $\mathscr{G}_x$ gives rise to a k-vector space $V = x^* \mathcal{F}$ with a $\operatorname{stab}(x)$-action. The bundle $\mathcal{F}$ is trivial, if and only if the representation V is trivial.

Quotients and sub-bundles of trivial ind-vector bundles on $\mathscr{G}_x$ are trivial. If x is special, then $\operatorname{stab}(x)$ is linearly reductive and extensions of trivial bundles are trivial.

Definition (3.1). Let $f \colon \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks with good moduli spaces. Let $x_0 \in |\mathscr{X}|$ be a (special) point and let $n \geq -1$ be an integer. We say that f is *n-$\mathbb{L}$-strong* at x_0 if the ind-vector bundle $\mathcal{H}^d(\mathbf{L}i_{x_0}^* \mathbb{L}_f)$ is trivial for every $d \geq -n$. Equivalently, if $x_0 \colon \operatorname{Spec} k \to \mathscr{X}$ is a representative of the point, then the $\operatorname{stab}(x_0)$-action on the k-vector space

$H^d(\mathbf{L}x_0^*\mathbb{L}_f)$ is trivial for every $d \geq -n$. We say that f is $\mathbb{L}$-*strong* at x_0 if f is n-$\mathbb{L}$-strong at x_0 for all n.

If f is locally of finite type, then $\mathcal{H}^0(\mathbf{L}i_{x_0}^*\mathbb{L}_f) = i_{x_0}^*\Omega_f$ is finite-dimensional. If f is locally of finite presentation, then one can show that $\mathcal{H}^{-1}(\mathbf{L}i_{x_0}^*\mathbb{L}_f)$ is finite-dimensional. We will not need this.

Lemma (3.2). *Let $x \in |X|$ be a point and let $x_0 \in |\mathscr{X}|$ be the corresponding special point.*

(i) *Let $n \in \{-1, 0, 1\}$. If f is n-strong at x, then f is n-$\mathbb{L}$-strong at x_0.*

(ii) *If f is strong at x, then f is 1-strong at x.*

(iii) *Assume that π_X and π_Y have affine diagonals and that either $\mathscr{X}$ is noetherian or f is locally of finite type. If f is flat and 0-strong at x, then f is $\mathbb{L}$-strong at x_0. In particular, f is 1-strong at x.*

Proof. Let $g\colon X \to Y$ denote the induced morphism on good moduli spaces.

(i) For $n = -1$, if $I_{\mathscr{X}} \to f^*I_{\mathscr{Y}}$ is injective at x_0, then f is relatively Deligne–Mumford in an open neighborhood of x_0 and hence $\mathbb{L}_f$ is concentrated in degrees ≤ 0 in that neighborhood. For $n = 0$, we have that $i_{x_0}^*\Omega_\rho = 0$ by Lemma 2.5 so $i_{x_0}^*(\pi_X)^*\Omega_g \to i_{x_0}^*\Omega_f$ is surjective which shows that the latter is trivial. For $n = 1$, we have that $\mathcal{H}^{-1}(\mathbf{L}i_x^*\mathbb{L}_f)$ is trivial by definition.

(ii) Strong morphisms are 0-strong so it is enough to prove that f is 1-$\mathbb{L}$-strong at x_0. We have a natural map $\mathbf{L}(\pi_X)^*\mathbb{L}_g \to \mathbb{L}_f$. Let E denote the cone of this map. If f is strong, that is, if $\mathscr{X} = \mathscr{Y} \times_Y X$, then E is the cotangent complex measuring the difference between the fiber product in derived algebraic geometry and classical algebraic geometry. The complex E is concentrated in degrees ≤ -2 [SP, 09DM]. It follows that

$$\mathcal{H}^d(\mathbf{L}i_{x_0}^*\mathbf{L}(\pi_X)^*\mathbb{L}_g) \to \mathcal{H}^d(\mathbf{L}i_{x_0}^*\mathbb{L}_f)$$

is an isomorphism for $d = 0$, that is, we have an equality of cotangent bundles, and surjective for $d = -1$. The result follows.

(iii) Since f is 0-strong, we have that $y_0 = f(x_0)$ is closed and we may assume that f is stabilizer preserving (Remark 2.8). Since f is flat, the cotangent complex commutes with arbitrary pull-back and we may assume that $\mathscr{Y} = \mathscr{G}_{y_0}$. Then $\mathscr{Y} \to Y$ is flat and f is strong (Lemma 2.3). By flat base change we then have that $\mathbb{L}_f = \mathbf{L}(\pi_X)^*\mathbb{L}_g$ and that $\mathbf{L}i_{x_0}^*\mathbb{L}_f$ is the pull-back of $\mathbf{L}i_x^*\mathbb{L}_g$ along $\mathscr{G}_{x_0} \to \operatorname{Spec}\kappa(x)$ and similarly for the dth cohomology sheaf. $\square$

Lemma (3.3). *Let $f\colon \mathscr{X} \to \mathscr{Y}$ and $g\colon \mathscr{Y} \to \mathscr{Z}$ be morphisms between algebraic stacks with good moduli spaces. Let $x_0 \in |\mathscr{X}|$ be a special point and $y_0 = f(x_0)$. Let $n \geq -1$ be an integer.*

(i) *If f is n-$\mathbb{L}$-strong at x_0 and g is n-$\mathbb{L}$-strong at y_0, then $g \circ f$ is n-$\mathbb{L}$-strong at x_0.*

(ii) *If $g \circ f$ is n-$\mathbb{L}$-strong at x and g is $(n-1)$-$\mathbb{L}$-strong at y_0, then f is n-$\mathbb{L}$-strong at x_0.*

Proof. The fundamental triangle for the composition $g \circ f$ gives the long exact sequence

$$\cdots \longrightarrow \mathcal{H}^{-(n+1)}(\mathbf{L}i_{x_0}^*\mathbb{L}_f) \longrightarrow$$
$$\longrightarrow \quad \mathcal{H}^{-n}(\mathbf{L}(f \circ i_{x_0})^*\mathbb{L}_g) \quad \longrightarrow \mathcal{H}^{-n}(\mathbf{L}i_{x_0}^*\mathbb{L}_{gf}) \longrightarrow \quad \mathcal{H}^{-n}(\mathbf{L}i_{x_0}^*\mathbb{L}_f) \quad \longrightarrow$$
$$\longrightarrow \mathcal{H}^{-(n-1)}(\mathbf{L}(f \circ i_{x_0})^*\mathbb{L}_g) \longrightarrow \qquad \cdots$$

Since the stabilizer of x_0 is linearly reductive, extensions of trivial ind-vector bundles on $\mathscr{G}_{x_0}$ are trivial. The result follows. $\qquad\square$

Proposition (3.4). *Let $f \colon \mathscr{X} \to \mathscr{Y}$ and $g \colon \mathscr{Y} \to \mathscr{Z}$ be morphisms between algebraic stacks with good moduli spaces. Assume that π_X, π_Y and π_Z have affine diagonals and that either $\mathscr{X}$ and $\mathscr{Y}$ are noetherian or that f and g are locally of finite type. Let $x \in |X|$ be a point and $y = f(x)$. Let $n \in \{0,1\}$.*

   (i) *If f is n-strong at x and g is n-strong at y, then $g \circ f$ is n-strong at x.*
   (ii) *If $g \circ f$ is n-strong at x and g is $(n-1)$-strong at y, then f is n-strong at x.*

Proof. When $n = 0$, the result follows directly from Lemma 2.5 and Proposition 2.6. For $n = 1$, the result then follows from Lemmas 3.2(i) and 3.3. $\quad\square$

4. Proof of the main theorem

In this section, we prove a local generalization of Theorem A and prove that all the listed properties of f descend to g except for the properties concerning local complete intersections and regular immersions that we defer to the following sections.

Lemma (4.1). *Let $f \colon \mathscr{X} \to \mathscr{Y}$ be a morphism between algebraic stacks with good moduli spaces. Assume that π_X and π_Y have affine diagonals. Further assume that f is 0-strong at $x \in |X|$. If f is locally of finite type at x_0, then $g \colon X \to Y$ is locally of finite type at x.*

Proof. After replacing X with an open neighborhood of x, we may assume that f is locally of finite type. After replacing X with a further open neighborhood, we can assume that $\rho \colon \mathscr{X} \to \mathscr{Y} \times_Y X$ is a closed immersion by Proposition 2.6. We may also assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine.

Write B as the union of its finitely generated subalgebras B_λ and let $X_\lambda = \operatorname{Spec} B_\lambda$. Since $\rho \colon \mathscr{X} \to \varprojlim_\lambda \mathscr{Y} \times_Y X_\lambda$ is a closed immersion and f is of finite type, it follows that $\rho_\lambda \colon \mathscr{X} \to \mathscr{Y} \times_Y X_\lambda$ is a closed immersion for sufficiently large λ. Hence the induced map on good moduli spaces $X \to X_\lambda$ is a closed immersion as well and we conclude that g is of finite type. $\quad\square$

Theorem (4.2). *Let $f \colon \mathscr{X} \to \mathscr{Y}$ be a morphism between algebraic stacks with good moduli spaces. Assume that π_X and π_Y have affine diagonals. Let $x \in |X|$ be a point. Assume that either $\mathscr{X}$ and $\mathscr{Y}$ are locally noetherian or that f is locally of finite presentation at x_0. Then the following are equivalent*

   (i) *f is 1-strong at x,*
   (ii) *f is strong at x.*

Proof. That (ii) $\Longrightarrow$ (i) is Lemma 3.2. For the converse, we can assume that X and Y are affine. After replacing X with an open neighborhood of x, we can assume that $\rho \colon \mathscr{X} \to \mathscr{Y} \times_Y X$ is a closed immersion (Proposition 2.6).

If $\mathscr{Y}$ is locally noetherian, then π_Y is of finite type so $\mathscr{Y} \times_Y X$ is noetherian and ρ is of finite presentation. If f is locally of finite presentation at x_0, then g is locally of finite type at x by Lemma 4.1 which implies that ρ is of finite presentation at x_0.

Moreover, ρ is 1-strong at x by Proposition 3.4. We can thus conclude that ρ, and hence f, is strong at x by Corollary 1.4. $\square$

That most properties in Theorem A descend follows from purity of π_Y.

Proposition (4.3). *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism between algebraic stacks with good moduli spaces. Suppose that f is strong. If f has one of the properties:*

   (i) *locally of finite type, locally of finite presentation, flat,*

   (ii) *syntomic, regular, smooth, étale, representable, monomorphism and locally of finite type, unramified, locally quasi-finite,*

  (iii) *affine, closed immersion, immersion, open immersion*

  (iv) *finite, proper, separated;*

then so does $g\colon X \to Y$.

Proof. Recall that $\pi_Y \colon \mathscr{Y} \to Y$ is *pure* (Section 1). If f is locally of finite type, locally of finite presentation, or flat, then so is g by descent along π_Y [SP, 08XD, 08XE].

The properties in (ii) are combinations of flat, locally of finite type, locally of finite presentation and conditions on the geometric fibers so they descend.

If f is affine, then so is g since $X = \operatorname{Spec}_Y(\pi_Y)_* f_* \mathcal{O}_{\mathscr{X}}$. If f is a closed immersion, then so is g since $(\pi_Y)_*$ is exact. If f is an immersion, let $x \in |X|$ and let U be an open neighborhood of $x_0 \in |\mathscr{Y}|$ such that $f|_U$ is a closed immersion. Then $g|_{Y \smallsetminus \pi_Y(\mathscr{Y} \smallsetminus U)}$ is a closed immersion. An open immersion is the same thing as an étale monomorphism.

If f is universally closed, then so is g since π_X and π_Y are surjective and universally closed. If f is separated, then Δ_f is a closed immersion and it follows that Δ_g is a closed immersion, that is, g is separated. We conclude that if f is proper, then so is g. Finite is equivalent to proper and affine and thus also descend (or use [SP, 08XD]). $\square$

Proof of Theorem A. Conditions (i)–(iii) says that f is 1-strong at every point $x \in |X|$. Thus, if they are satisfied, then f is strong by Theorem 4.2. Conversely, if f is strong, then f is special and stabilizer preserving and also 1-$\mathbb{L}$-strong at every special point by Lemma 3.2(ii).

If f has one of the properties of (a)–(c) then so does g by Propositions 4.3 (most properties), 5.1 (quasi-regular and Koszul-regular) and 6.1 (lci).

If π_Y is a coarse moduli space, then every point of $\mathscr{Y}$ is special so (i) is redundant. If f has reduced fibers, then fiberwise stabilizer preserving at $y_0 \in |\mathscr{Y}|$ is equivalent to pointwise stabilizer preserving at every point of $f^{-1}(y_0)$ by Lemma 2.2. Conditions (i)–(ii) says that f is 0-strong at every point $x \in |X|$. If in addition f is flat, then f is 1-strong at every point $x \in |X|$ by Lemma 3.2(iii). $\square$

Proof of Corollary C. It is immediate that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv). If $g_1, g_2 \colon X \to \mathscr{Y}$ are two maps through which f factors, then the we obtain a map $\tau \colon \mathscr{X} \to \mathscr{Y} \times_{\mathscr{Y} \times \mathscr{Y}} X$. Since π is initial among algebraic spaces,

τ factors uniquely as $\mathscr{X} \to X \to \mathscr{Y} \times_{\mathscr{Y} \times \mathscr{Y}} X$, that is, we have a unique 2-isomorphism between g_1 and g_2.

To see that (iv) $\implies$ (i), we may thus work fppf-locally on X. Let $U \to \mathscr{Y}$ be a smooth presentation and let $\mathscr{X}' = \mathscr{X} \times_{\mathscr{Y}} U$. Then $\mathscr{X}' \to \mathscr{X}$ is smooth, surjective and fiberwise stabilizer preserving by (iv). By Theorem A, it thus descends to a smooth surjective morphism $X' \to X$. Since $\mathscr{X}' \to X'$ is a coarse moduli space, we obtain a factorization $\mathscr{X}' \to X' \to U$ which gives a map $X' \to U \to \mathscr{Y}$ as requested. $\qquad\square$

5. Regular immersions

In this section, we prove the part about regular immersions in Theorem A. We begin with recalling various notions of regular sequences and regular immersions in the non-noetherian situation. The reader who is only interested in the noetherian situation can skip ahead to §5.4.

5.1. **Regular sequences.** Let A be a ring, let $f_1, f_2, \ldots, f_n \in A$ be a sequence of elements and let $I = (f_1, f_2, \ldots, f_n)$. There are three slightly different notions of $f_1, f_2, \ldots, f_n$ being a regular sequence [EGA$_{\text{IV}}$, 0.15, 16.9, 19.5], [SGA$_6$, Exp. VII], [SP, 062D, 07CU, 067M].

  (i) f_i is a non-zero divisor in $A/(f_1, \ldots, f_{i-1})$ for all $i = 1, 2, \ldots, n$.
  (ii) The Koszul complex $K_\bullet(A, f_1, f_2, \ldots, f_n)$ is acyclic in degrees > 0.
  (iii) I/I^2 is locally free of rank n and the canonical map $\operatorname{Sym}_{A/I}(I/I^2) \to \bigoplus_{d \geq 0} I^d/I^{d+1}$ is an isomorphism.

We say that the sequence (f_i) is *regular* in (i), *Koszul-regular* in (ii) and *quasi-regular* in (iii). If $(f_1, f_2, \ldots, f_n) = (g_1, g_2, \ldots, g_n)$, then $\{f_i\}$ is Koszul-regular if and only if $\{g_i\}$ is Koszul-regular [SP, 066A]. This is also trivially true for quasi-regularity but false for regular sequences in general [EGA$_{\text{IV}}$, Rmq. 16.9.6 (ii)].

5.2. **Schemes.** Let X be a scheme and let $f \colon Z \hookrightarrow X$ be a closed immersion with corresponding sheaf of ideals $\mathcal{I}$. We say that f is a regular (resp. a Koszul-regular, resp. a quasi-regular) immersion at $x \in |X|$ if there exists an affine open neighborhood $U = \operatorname{Spec} A \subseteq X$ of x such that $\mathcal{I}$ is generated by a regular (resp. Koszul-regular, resp. quasi-regular) sequence. We have that regular $\implies$ Koszul-regular $\implies$ quasi-regular. When X is locally noetherian, all three notions coincide, see [EGA$_{\text{IV}}$, Prop. 16.9.11, Cor. 19.5.2] or [SP, 063E, 063I]. Koszul-regularity and quasi-regularity are easily seen to be fpqc-local [SP, 068N] whereas regularity is not. Indeed, any Koszul-regular immersion is smooth-locally regular [SP, 068Q].

5.3. **Algebraic stacks.** We say that a closed immersion $f \colon \mathscr{Z} \hookrightarrow \mathscr{X}$ of algebraic stacks is Koszul-regular (resp. quasi-regular) at a point $x \in |\mathscr{X}|$ if there exists a scheme U, a smooth morphism $U \to \mathscr{X}$ and a point $u \in |U|$ above x, such that $\mathscr{Z} \times_{\mathscr{X}} U \to U$ is Koszul-regular (resp. quasi-regular) at u. This then holds for any U and any point u above x.

Let $f \colon \mathscr{Z} \hookrightarrow \mathscr{X}$ be a closed immersion, let $x \in |\mathscr{X}|$ be a point and let $i_x \colon \mathscr{G}_x \hookrightarrow \mathscr{X}$ denote the inclusion of the residual gerbe (or a gerbe representative as in Remark 1.7). Consider the following conditions.

  (i) f is Koszul-regular at x,

(ii) $\mathcal{I}/\mathcal{I}^2$ is locally free at x and $\mathbb{L}_f \simeq (\mathcal{I}/\mathcal{I}^2)[1]$ in a neighborhood of x.

(iii) $\mathcal{H}^{-n}(\mathbf{L}i_x^*\mathbb{L}_f) = 0$ for $n \geq 2$, and

(iv) $\mathcal{H}^{-n}(\mathbf{L}i_x^*\mathbb{L}_f) = 0$ for $n = 2$.

Then (i) $\Longrightarrow$ (ii) [SP, 08SK] and trivially (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv). If $\mathscr{X}$ is locally noetherian, then all conditions are equivalent, see [And74, Thm. 6.25].

5.4. **Descent of regular immersions.** Recall that a closed immersion $f\colon \mathscr{X} \to \mathscr{Y}$ with ideal sheaf $\mathcal{I}$ is a quasi-regular immersion, if

(i) $\mathcal{I}$ is finitely generated (i.e., f is finitely presented),

(ii) the conormal sheaf $\mathcal{N}_f := \mathcal{I}/\mathcal{I}^2 = f^*\mathcal{I}$ is a vector bundle on $\mathscr{X}$, and

(iii) the natural map $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{N}_f) \to \bigoplus_{d \geq 0} \mathcal{I}^d/\mathcal{I}^{d+1}$ is an isomorphism.

Proposition (5.1). *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a closed immersion between stacks with good moduli spaces and let $g\colon X \to Y$ denote the induced closed immersion of good moduli spaces. Suppose that f is strong.*

(i) *If f is a quasi-regular immersion, then so is g.*

(ii) *If f is Koszul-regular, then so is g.*

In either case, the adjunction maps $(\pi_X)^*\mathcal{N}_g \to \mathcal{N}_f$ and $\mathcal{N}_g \to (\pi_X)_*\mathcal{N}_f$ are isomorphisms.

Proof. Let $\mathcal{I}$ denote the ideal sheaf of f and $\mathcal{J}$ the ideal sheaf of g. Since f is strong, we have that $(\pi_Y)^*\mathcal{J}^d \to \mathcal{I}^d$ is surjective for all $d \geq 0$. Thus, since f is cohomologically affine, $\mathcal{J}^d = (\pi_Y)_*(\mathcal{I}^d)$ for all $d \geq 0$ and hence $\mathcal{J}^d/\mathcal{J}^{d+1} = (\pi_X)_*(\mathcal{I}^d/\mathcal{I}^{d+1})$. In particular,

$$\mathcal{N}_g = (\pi_X)_*\mathcal{N}_f, \quad \text{and} \quad \bigoplus_{d \geq 0} \mathcal{J}^d/\mathcal{J}^{d+1} = (\pi_X)_* \bigoplus_{d \geq 0} \mathcal{I}^d/\mathcal{I}^{d+1}.$$

If $\mathcal{N}_f$ is a vector bundle, then so is $(\pi_X)_*\mathcal{N}_f$ by Theorem 1.3(iii) applied to $\mathcal{F} = \mathcal{N}_f = f^*\mathcal{I}$ along π_X, and $(\pi_X)^*(\pi_X)_*\mathcal{N}_f \to \mathcal{N}_f$ is an isomorphism. It follows that $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{N}_g) = (\pi_X)_* \mathrm{Sym}_{\mathcal{O}_{\mathscr{X}}}(\mathcal{N}_f)$. Thus, if f is quasi-regular, then so is g.

Now, suppose that f is Koszul-regular. To see that g is Koszul-regular we may work locally on Y and assume that $Y = \mathrm{Spec}\, A$ is affine. Choose a sequence of elements $f_1, f_2, \ldots, f_n \in \mathcal{J}$ that gives a basis of $\mathcal{N}_g$. After replacing Y with an open neighborhood, we may assume that $\mathcal{J} = (f_1, f_2, \ldots, f_n)$. Then $(\pi_Y)^*f_1, \ldots, (\pi_Y)^*f_n \in \mathcal{I}$ generates $\mathcal{I}$ and gives a basis of $\mathcal{N}_f = (\pi_X)^*\mathcal{N}_g$. Consider the Koszul complex $K_\bullet = K_\bullet(A, f_1, f_2, \ldots, f_n)$. Since f is Koszul-regular, the pull-back $(\pi_Y)^*K_\bullet$ is acyclic in degrees > 0. Since $(\pi_Y)_*$ is exact, the push-forward $(\pi_Y)_*(\pi_Y)^*K_\bullet = K_\bullet$ is also acyclic in degrees > 0 and we conclude that g is Koszul-regular. $\square$

Remark **(5.2).** *cf.* [Edi16, Thm. 2.2 (ii)] — Suppose that $\mathscr{X}$ is noetherian (or merely quasi-compact and quasi-separated). If f is a strong quasi-regular immersion, then there exists a finite stratification $X = \bigcup X_k$ by locally closed subspaces such that $(\mathcal{N}_f)|_{\mathscr{X} \times_X X_k}$ is a trivial vector bundle for every k. Indeed, there exists a finite stratification of X in schemes. After refining the stratification we may assume that $\mathcal{N}_g|_{X_k}$ is a trivial vector bundle for every k and the result follows.

6. Local complete intersection morphisms

In this section, we prove the part about lci morphisms in Theorem A. We begin with recalling the definition of lci morphisms $f\colon \mathscr{X} \to \mathscr{Y}$ when either (a) $\mathscr{X}$ and $\mathscr{Y}$ are noetherian schemes or stacks but f is not locally of finite type, or (b) $\mathscr{X}$ and $\mathscr{Y}$ are not necessarily noetherian but f is locally of finite presentation. The reader who only is interested in morphisms of finite type between noetherian stacks can skip ahead to §6.6. Main references for this section are [Avr99], [SP, 068E] and [EGA$_{\mathrm{IV}}$, 19.3].

6.1. Algebraic schemes.
Let X be a scheme of finite type over a field k. We say that X is a *local complete intersection at* x, abbreviated lci, if locally around x, there is a closed immersion $i\colon X \hookrightarrow Y$ where Y is smooth over k and i is regular at x. If X is lci at x, then i is regular at x for any such factorization. Equivalently, $H^2(\mathbb{L}_{X/k} \otimes \kappa(x)) = 0$ and then $H^{-n}(\mathbb{L}_{X/k} \otimes \kappa(x)) = 0$ for all $n \geq 2$.

6.2. Morphisms locally of finite presentation.
If $f\colon \mathscr{X} \to \mathscr{Y}$ is a morphism, locally of finite presentation, between algebraic stacks, then we say that f is lci at $x \in |\mathscr{X}|$ if there exists a commutative diagram

$$
\begin{array}{ccc}
U & \overset{i}{\hookrightarrow} & V \\
\downarrow & & \downarrow \\
\mathscr{X} & \overset{f}{\longrightarrow} & \mathscr{Y}
\end{array}
$$

where U and V are schemes, the vertical maps are smooth and i is a closed immersion such that i is Koszul-regular at a point u above x. Then i is Koszul-regular for any such diagram [SP, 0692, 069P].

A closed immersion is Koszul-regular if and only if it is lci. Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism, locally of finite presentation, let $x \in |\mathscr{X}|$ and consider the following conditions:

(i) f is lci at x,
(ii) $\mathbb{L}_f|_U$ is perfect of Tor-amplitude $[-1,1]$ in an open neighborhood U of x.
(iii) $\mathcal{H}^{-n}(\mathbf{L}i_x^*\mathbb{L}_f) = 0$ for $n \geq 2$, and
(iv) $\mathcal{H}^{-n}(\mathbf{L}i_x^*\mathbb{L}_f) = 0$ for $n = 2$.

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) and all conditions are equivalent if $\mathscr{X}$ is locally noetherian. This is an easy consequence of the corresponding result for Koszul-regular immersions using that the cotangent complex of a smooth morphism is perfect of Tor-amplitude $[0,1]$.

6.3. Noetherian schemes.
Similar to regularity, there is an *absolute* notion of lci: if X is a noetherian scheme, then we say that X is lci at x if the completion $\widehat{\mathcal{O}}_{X,x}$ is a quotient of a regular local ring R by a regular sequence. If we write $\widehat{\mathcal{O}}_{X,x}$ as a quotient R/I of some regular ring R, which we always can do by Cohen's structure theorem, then I is generated by a regular sequence if and only if X is lci at x.

6.4. **Morphisms between noetherian schemes and stacks.** Let $f\colon X \to Y$ be a morphism of locally noetherian schemes. A *Cohen factorization* of f at $x \in |X|$ is a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec}\widehat{\mathcal{O}}_{X,x} & \xhookrightarrow{\ i\ } & W \\
\downarrow & & \downarrow{\scriptstyle w} \\
X & \xrightarrow[f]{} & Y
\end{array}
$$

where w is flat, the fiber $w^{-1}(f(x))$ is a regular scheme, and i is a closed immersion. Cohen factorizations always exist [AFH94, Thm. 1.1].

A morphism $f\colon X \to Y$ of locally noetherian schemes is lci at $x \in |X|$ if i is regular for some Cohen factorization. This does not depend on the Cohen factorization and the following are equivalent [Avr99, Thm. 1.2, (1.8)]

  (i) f is lci at x,
  (ii) $H^{-n}(\mathbb{L}_f \overset{\mathbf{L}}{\otimes} \kappa(x)) = 0$ for $n \geq 2$, and
  (iii) $H^{-n}(\mathbb{L}_f \overset{\mathbf{L}}{\otimes} \kappa(x)) = 0$ for $n = 2$.

The latter characterizations show that the notion of lci is local in the smooth topology on both X and Y and hence makes sense for morphisms of locally noetherian algebraic stacks. To make sense of the conditions above for stacks, replace $H^{-n}(\mathbb{L}_f \overset{\mathbf{L}}{\otimes} \kappa(x))$ with $\mathcal{H}^{-n}(\mathbf{L}i_x^*\mathbb{L}_f)$.

6.5. **Syntomic morphisms.** Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism between algebraic stacks. Assume that either $\mathscr{X}$ and $\mathscr{Y}$ are noetherian or that f is locally of finite presentation. Then we say that f is *syntomic* at $x \in |X|$ if it is flat and lci at x. Equivalently, f is flat at x and the fiber $f^{-1}(f(x))$ is lci at x. Indeed, in the noetherian case this follows from the characterization using the cotangent complex since the cotangent complex $\mathbb{L}_f$ commutes with arbitrary base change since f is flat. When f is locally of finite presentation, this is [EGA$_{\mathrm{IV}}$, Prop. 19.3.7] and then the locus where f is syntomic is open in X.

6.6. **Descent of lci morphisms.**

Proposition (6.1). *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism between stacks with good moduli spaces and let $g\colon X \to Y$ denote the induced morphism of good moduli spaces. Assume that either f is locally of finite presentation or that $\mathscr{X}$ and $\mathscr{Y}$ are locally noetherian. Suppose that f is strong. If f is lci, then so is g.*

Proof. First assume that f is locally of finite presentation. Then so is g (Proposition 4.3). The question is local on X and Y so we may assume that X and Y are affine and can factor g through a closed immersion $X \hookrightarrow \mathbb{A}_Y^n$. Then g is lci if and only if $X \hookrightarrow \mathbb{A}_Y^n$ is Koszul-regular and f is lci if and only if $\mathscr{X} \hookrightarrow \mathbb{A}_{\mathscr{Y}}^n$ is Koszul-regular (see §6.2). The result now follows by Proposition 5.1.

Instead assume that $\mathscr{X}$ and $\mathscr{Y}$ are locally noetherian. Let $x \in |X|$ be a point, let $y = g(x)$, let $\widehat{X}$ denote the completion at x and let $\widehat{X} \hookrightarrow W \to Y$ be a Cohen factorization [AFH94, Thm. 1.1]. That is, $W \to Y$ is flat, the

fiber W_y is regular and $\widehat{X} \hookrightarrow W$ is a closed immersion. Let $\widehat{\mathscr{X}} \hookrightarrow \mathscr{W} \to \mathscr{Y}$ be the pull-back of the Cohen factorization along π_Y. Note that $\mathscr{W}_{y_0} = \mathscr{W} \times_{\mathscr{Y}} \mathscr{G}_{y_0}$ is regular. Indeed, the morphism $\mathscr{G}_{y_0} \to \operatorname{Spec} \kappa(y)$ is smooth since it is an fppf gerbe. If f is lci, we conclude that $\widehat{\mathscr{X}} \hookrightarrow \mathscr{W}$ is a regular immersion and that $\widehat{X} \hookrightarrow W$ is regular (Proposition 5.1) so that g is lci. $\quad\square$

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